

EXPANDING THURSTON MAPS AS QUOTIENTS

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ABSTRACT. A *Thurston map* is a branched covering map $f: S^2 \rightarrow S^2$ that is *postcritically finite*. *Mating of polynomials*, introduced by Douady and Hubbard, is a method to *geometrically* combine the Julia sets of two polynomials (and their dynamics) to form a rational map. We show that every *expanding* Thurston map f has an iterate $F = f^n$ that is obtained as the mating of two polynomials. One obtains a concise description of F via *critical portraits*. The proof is based on the construction of the invariant Peano curve from [Mey]. As another consequence we obtain a large number of fractal tilings of the plane and the hyperbolic plane.

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1. INTRODUCTION

A *Thurston map* is a branched covering of the sphere $f: S^2 \rightarrow S^2$ that is *post-critically finite*. Thurston has given a *topological characterization* of rational maps among Thurston maps (see [DH93]). We consider such maps that are *expanding* in a suitable sense (see Section 2.1 for precise definitions).

The paper present is a direct continuation of [Mey], where the following theorem was proved.

Theorem 1.1. *Let f be a Thurston map. Then f is expanding if and only if there is an iterate $F = f^n$, a Peano curve $\gamma: S^1 \rightarrow S^2$ (onto) such that $F(\gamma(z)) = \gamma(z^d)$. Here $d = \deg F$. This means that the following diagram commutes.*

$$\begin{array}{ccc}
 S^1 & \xrightarrow{z^d} & S^1 \\
 \gamma \downarrow & & \downarrow \gamma \\
 S^2 & \xrightarrow{F} & S^2
 \end{array}$$

Here further implications of this theorem are investigated. Note that we do not only use the theorem, but the precise construction of γ . The main inspiration continues to be Milnor's paper [Mil04]. There, an invariant Peano curve γ as above was constructed for one specific example F , with implications analogous to the ones below.

Theorem 1.2. *The map $\gamma: S^1 \rightarrow S^2$ maps normalized Lebesgue measure on S^1 to the measure of maximal entropy on S^2 with respect to F .*

By “normalized” is meant that the total mass is 1, i.e., that the measure is a probability measure. The *measure of maximal entropy* (also called the *Lyubich* or *Brolin* measure) is the unique invariant probability measure that maximizes the (measure theoretic) entropy. It can be defined as the weak limit of $1/d^n \sum_{y \in F^{-n}(x_0)} \delta_y$, for any point $x_0 \in S^2$. Note that the measure of maximal entropy of $F = f^n$ equals the measure of maximal entropy of f .

1.1. Describing F via critical portraits. Using the invariant Peano curve $\gamma: S^1 \rightarrow S^2$ from Theorem 1.1, an *equivalence relation* on S^1 is defined by

$$(1.1) \quad s \sim t \Leftrightarrow \gamma(s) = \gamma(t),$$

for all $s, t \in S^1$. Elementary topology yields that S^1/\sim is homeomorphic to S^2 and that $z^d/\sim: S^1/\sim \rightarrow S^1/\sim$ is topologically conjugate to the map F .

Theorem 1.3. *The following diagram commutes,*

$$\begin{array}{ccc} S^1/\sim & \xrightarrow{z^d/\sim} & S^1/\sim \\ h \downarrow & & \downarrow h \\ S^2 & \xrightarrow{F} & S^2. \end{array}$$

Here the homeomorphism $h: S^1/\sim \rightarrow S^2$ is given by $h: [s] \mapsto \gamma(s)$, for all $s \in S^1$.

The equivalence relation (1.1) may be constructed from *finite data*, more precisely from two finite families of finite sets of rational numbers. Thus these families encode the map F up to topological conjugacy by the previous theorem.

The proper setting is as follows. The Peano curve γ from Theorem 1.1 was constructed as the limit of *approximations* γ^n . The approximations have finitely many points where they touch themselves, but they never cross themselves. Thus $S^2 \setminus \gamma^n$ has a *white* and a *black* component. We define equivalence relations $\sim^{n,w}, \sim^{n,b}$ on S^1 by $s \sim^{n,w} t$ ($s \sim^{n,b} t$) whenever $\gamma^n(s) = \gamma^n(t)$ and γ^n touches itself at s, t in the white component (black component). It turns out that

- \sim can be recovered from the equivalence relations $\sim^{n,w}, \sim^{n,b}$ as a limit (defined in a suitable sense, see Theorem 4.7).
- The equivalence relations $\sim^{n,w}, \sim^{n,b}$ can be *inductively obtained* from the “initial ones” $\sim^{1,w}, \sim^{1,b}$ (Theorem 5.10).
- The equivalence classes of $\sim^{1,w}, \sim^{1,b}$ form a *critical portrait* in the sense of [Poi93], see Definition 5.12. Such a critical portrait is a (finite) family of finite sets of rational angles.

Thus we obtain the following first main result of this paper. The map $F = f^n$ is the (same) iterate of the expanding Thurston map f from Theorem 1.1.

Theorem 1.4. *F admits a description via two critical portraits.*

This provides an effective way to describe F .

1.2. The Carathéodory semi-conjugacy of a polynomial Julia set. The description of F as in Theorem 1.3 may be viewed as a two-sided version of the viewpoint introduced by Douady-Hubbard and Thurston ([DH84], [DH85], [Thu85], see also [Ree92] and [Kel00]), namely the combinatorial description of Julia sets in terms of *external rays*.

Let P be a monic polynomial (i.e., the coefficient of the leading term is 1) with connected and locally connected filled Julia set \mathcal{K} . Let $\phi: \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \rightarrow \widehat{\mathbb{C}} \setminus \mathcal{K}$ be the Riemann map normalized by $\phi(\infty) = \infty$ and $\phi'(\infty) = \lim_{z \rightarrow \infty} \phi(z)/z > 0$ (in fact then $\phi'(\infty) = 1$). By Carathéodory's theorem (see for example [Mil99, Theorem 17.14]) ϕ extends continuously to

$$(1.2) \quad \sigma: S^1 = \partial \overline{\mathbb{D}} \rightarrow \partial \mathcal{K} = \mathcal{J},$$

where \mathcal{J} is the *Julia set* of P . We call the map σ the *Carathéodory semi-conjugacy* of \mathcal{J} . We remind the reader that every postcritically finite polynomial has connected and locally connected filled Julia set (see for example [Mil99, Theorem 19.7]).

Consider the equivalence relation on S^1 induced by the Carathéodory semi-conjugacy, namely

$$(1.3) \quad s \approx t :\Leftrightarrow \sigma(s) = \sigma(t),$$

for all $s, t \in S^1$. Elementary topology (see Lemma 4.6) yields that S^1/\approx is homeomorphic to \mathcal{J} , where the homeomorphism is given by $h: S^1/\approx \rightarrow \mathcal{J}$, $[s]_{\approx} \mapsto \sigma(s)$. Böttcher's theorem (see for example [Mil99, § 9]) says that the Riemann map ϕ conjugates z^d to the polynomial P on $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, where $d = \deg P$. This implies that the following diagram commutes,

$$(1.4) \quad \begin{array}{ccc} S^1/\approx & \xrightarrow{z^d/\approx} & S^1/\approx \\ \downarrow h & & \downarrow h \\ \mathcal{J} & \xrightarrow{P} & \mathcal{J}. \end{array}$$

Clearly Theorem 1.3 (and thus Theorem 1.1) corresponds to the above. The connection however is deeper, namely F is obtained from the “mating of two polynomials” via their Carathéodory semi-conjugacies.

1.3. Mating of polynomials. Douady and Hubbard [Dou83] (see also [Mil04]), introduced the construction of *mating of polynomials* as a way to geometrically combine two polynomials to form a rational map. The main result of the paper present is that some iterate of each expanding Thurston map is obtained in this way. We recall the construction briefly, before stating the main theorems precisely.

It will be convenient to identify the unit circle S^1 with \mathbb{R}/\mathbb{Z} . We still write

$$(1.5) \quad \sigma: \mathbb{R}/\mathbb{Z} \rightarrow \mathcal{J} = \partial \mathcal{K}$$

for the Carathéodory semi-conjugacy.

Consider two monic polynomials P_w, P_b (called the *white* and the *black* polynomial) of the same degree with connected and locally connected Julia sets. Let σ_w, σ_b be the Carathéodory semi-conjugacies of their Julia sets $\mathcal{J}_w, \mathcal{J}_b$.

Glue the filled Julia sets $\mathcal{K}_w, \mathcal{K}_b$ (of P_w, P_b) together by identifying $\sigma_w(t) \in \partial \mathcal{K}_w$ with $\sigma_b(-t) \in \partial \mathcal{K}_b$. More precisely, we consider the disjoint union of $\mathcal{K}_w, \mathcal{K}_b$, and let $\mathcal{K}_w \amalg \mathcal{K}_b$ be the quotient obtained from the equivalence relation generated by

$\sigma_w(t) \sim \sigma_b(-t)$ for all $t \in \mathbb{R}/\mathbb{Z}$. The minus sign is customary here, though not essential: identifying $\sigma_w(t)$ with $\sigma_b(t)$ amounts to the mating of P_w with $\overline{P_b(\bar{z})}$. The *topological mating* of P_w, P_b is the map

$$P_w \amalg P_b: \mathcal{K}_w \amalg \mathcal{K}_b \rightarrow \mathcal{K}_w \amalg \mathcal{K}_b,$$

given by

$$P_w \amalg P_b|_{\mathcal{K}_i} = P_i,$$

for $i = w, b$. It follows from (1.4) that it is well defined, namely that $x_w \sim x_b \Rightarrow P_w(x_w) \sim P_b(x_b)$ (for all $x_w \in \mathcal{K}_w, x_b \in \mathcal{K}_b$). If a map is topologically conjugate to a map $P_w \amalg P_b$ we say it is obtained as a (topological) mating.

Recall that a *periodic critical point* (of a Thurston map f) is a critical point c , such that $f^k(c) = c$ for some $k \geq 1$. The following is the second main result of this paper. It covers the case when f is a postcritically finite rational map, that has the whole sphere as its Julia set.

Theorem 1.5. *For every expanding Thurston map f without periodic critical points there is an iterate $F = f^n$, that is obtained as a topological mating of two polynomials P_w, P_b .*

The polynomials P_w, P_b are postcritically finite, where each critical point is *strictly* preperiodic (i.e., P_w, P_b have no periodic critical points). The iterate $F = f^n$ is the same as the one from Theorem 1.1.

1.4. Identifying Fatou components. Theorem 1.5 is actually a special case of the following more general theorem. Here f is allowed to have periodic critical points. In this case, clearly no iterate $F = f^n$ can be the mating of two polynomials.

It is however possible to slightly alter the construction. Namely identify the closure of each bounded Fatou component of P_w and P_b . In addition we need to take the *closure* of the equivalence relation. An equivalence relation \sim on a compact metric space S is called *closed* if it is closed as a subset of the product space $S \times S$. If \sim is not closed the quotient S/\sim fails to be Hausdorff.

Formally we consider the equivalence relation on the disjoint union of $\mathcal{K}_w, \mathcal{K}_b$ generated by the following,

$$\begin{aligned} \sigma_w(t) &\sim \sigma_b(-t), \quad \text{for all } t \in \mathbb{R}/\mathbb{Z} \text{ and} \\ x &\sim y, \quad \text{if } x, y \in \text{clos } \mathcal{F}_w \text{ or } x, y \in \text{clos } \mathcal{F}_b, \end{aligned}$$

for all $x, y \in \mathcal{K}_w$, or $x, y \in \mathcal{K}_b$. Here \mathcal{F}_w is a bounded component of the Fatou set of P_w and \mathcal{F}_b a bounded component of the Fatou set of P_b . Since \sim may not be closed in general, we consider the closure $\hat{\sim}$ of \sim (see Lemma 4.5). Let $\mathcal{K}_w \hat{\amalg} \mathcal{K}_b$ be the quotient of (the disjoint union of) $\mathcal{K}_w, \mathcal{K}_b$ with $\hat{\sim}$. We will show that the maps P_w, P_b descend to this quotient, meaning that the following is well defined.

$$(1.6) \quad \begin{aligned} P_w \hat{\amalg} P_b: \mathcal{K}_w \hat{\amalg} \mathcal{K}_b &\rightarrow \mathcal{K}_w \hat{\amalg} \mathcal{K}_b, \\ P_w \hat{\amalg} P_b([x]) &:= \begin{cases} [P_w(x)], & x \in \mathcal{K}_w; \\ [P_b(x)], & x \in \mathcal{K}_b. \end{cases} \end{aligned}$$

Theorem 1.6. *Let f be an expanding Thurston map. Then an iterate $F = f^n$ is topologically conjugate to a map $P_w \hat{\amalg} P_b$ as above.*

The polynomials P_w, P_b are postcritically finite, furthermore their Fatou sets are *separated*. This means that two distinct bounded Fatou components of P_w (of P_b) have disjoint closures. Again the iterate $F = f^n$ is the same as the one from Theorem 1.1.

1.5. Outline. In Section 2 we recall the setup from [BM]. Namely one picks a Jordan curve \mathcal{C} containing all postcritical points. Then $F^{-n}(\mathcal{C})$ decomposes the sphere S^2 into n -tiles. This in turn allows for a combinatorial description of F .

In Section 3 the *construction* of the invariant Peano curve γ from Theorem 1.1 as given in [Mey] is *reviewed*. In particular γ was constructed as a limit of *approximations* γ^n , whose properties we list.

Some elementary facts about equivalence relations are provided in Section 4.

In Section 5 we introduce *equivalence relations* $\overset{n,w}{\sim}, \overset{n,b}{\sim}$. They describe the self-intersections of the approximations γ^n . It is shown that these equivalence relations are obtained inductively, i.e., from $\overset{1,w}{\sim}, \overset{1,b}{\sim}$. These “initial equivalence relations” $\overset{1,w}{\sim}, \overset{1,b}{\sim}$ form a *critical portrait* in the sense of [Poi93]. Furthermore the map F is completely determined from them, up to topological conjugacy.

In Section 6 we investigate the *sizes* of the *equivalence classes* induced by γ . More precisely we show that if F does not have periodic critical points, the size of such equivalence classes is bounded by some number $N < \infty$. If F has periodic critical points, we show that at least one equivalence class is finite.

In Section 7 we show that the invariant Peano curve γ maps *Lebesgue measure* on the circle S^1 to the *measure of maximal entropy* of F (on S^2), i.e., prove Theorem 1.2.

In Section 8 we show that F is *obtained as a mating*, for the case when F has no periodic critical points; i.e., Theorem 1.5 is proved.

The case when F has periodic critical points, or Theorem 1.6, is proved in Section 9.

In Section 10 we illustrate the fractal tilings obtained from the construction. This also shows explicitly the invariant Peano curve for some examples.

We conclude the paper in Section 11 with some open questions.

1.6. Notation. The circle is denoted by S^1 , the 2-sphere by S^2 . By $\text{int } U$ we denote the interior, by $\text{clos } U$ the closure of a set U . The cardinality of a (finite) set S is denoted by $\#S$.

It will often be convenient to identify S^1 with \mathbb{R}/\mathbb{Z} ; the map $z^d: S^1 \rightarrow S^1$ then is denoted by

$$\mu = \mu_d: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}, \quad \mu(t) = dt(\text{mod } 1),$$

the n -th iterate is $\mu^n(t) = d^n t(\text{mod } 1)$.

For two non-negative expressions A, B we write $A \lesssim B$ if there is a constant $C > 0$ such that $A \leq CB$. We refer to C as $C(\lesssim)$. Similarly we write $A \asymp B$ if $A/C \leq B \leq CA$ for a constant $C \geq 1$, we refer to C as $C(\asymp)$.

- The n -iterate of a map f is denoted by f^n .
- $F = f^n$ is the iterate of the expanding Thurston map f from Theorem 1.1.
- By $\text{crit} = \text{crit}(f)$, $\text{post} = \text{post}(f)$ we denote the *set of critical* and *postcritical points* (see Section 2.1).
- The *degree* of F is denoted by d , the *number of postcritical points* by k .
- The *local degree* of F at $v \in S^2$ is denoted by $\deg_F(v)$, see Definition 2.1.

- *Upper indices* indicate the *order* of an object. For example some preimage of some object U^0 by F^n will be denoted by U^n . Also maps and other objects associated with such objects U^n will have an upper index “ n ”.
- \mathcal{C} is a Jordan curve containing all postcritical points.
- X_w^0, X_b^0 are the white/black 0-tiles, i.e., the closures of the two components of $S^2 \setminus \mathcal{C}$.
- *Lower indices* “ w ” or “ b ” indicate whether an object is colored “white” or “black”, i.e., if it is mapped eventually to X_w^0 or X_b^0 ; or closely related to such objects.
- The visual metric is denoted by $|x - y|_{\mathcal{S}}$, see Section 2.3.
- The *sets of all n -tiles, -edges, -vertices* are denoted by $\mathbf{X}^n, \mathbf{E}^n, \mathbf{V}^n$ (Section 2.2).
- γ^n is the n -th approximation of the Peano curve γ (Section 3.1).
- A point $\alpha^n \in S^1$ such that $\gamma^n(\alpha^n) \in \mathbf{V}^n$ is called an n -angle. The *set of all n -angles* is denoted by \mathbf{A}^n (Section 3.1).
- An n -arc a^n is a closed interval in $\mathbb{R}/\mathbb{Z} = S^1$ that is mapped by γ^n (homeomorphically) to an n -edge.
- H^n are the *pseudo-isotopies* from which the approximations γ^n were constructed, see Section 3.2.
- $\pi_w \cup \pi_b$ is a cnc-partition (of a set $\{0, 1, \dots, 2n-1\}$). It describes the connection at a vertex, i.e., which white/black tiles are “connected” at v (see Section 3.3).
- By \sim we denote the equivalence relation on S^1 induced by the invariant Peano curve ($s \sim t \Leftrightarrow \gamma(s) = \gamma(t)$), by $\overset{n}{\sim}$ the equivalence relation induced by γ^n (Section 4.3).
- The *join* of two equivalence relations $\overset{a}{\sim}, \overset{b}{\sim}$ is denoted by $\overset{a}{\sim} \vee \overset{b}{\sim}$, their *meet* by $\overset{a}{\sim} \wedge \overset{b}{\sim}$ (Section 4.1).
- The *equivalence relations* $\overset{n,w}{\sim}, \overset{n,b}{\sim}$ describe where γ^n “touches itself on the white/black side” (see Definition 5.1). Their equivalence classes are denoted by $[\alpha]_{n,w}, [\alpha]_{n,b}$.
- S_w^2, S_b^2 are the *white/black hemispheres*, i.e., the components of $S^2 \setminus S^1$. They are equipped with the hyperbolic metric.
- $\mathcal{L}_w^n, \mathcal{L}_b^n$ are the *laminations* associated to $\overset{n,w}{\sim}, \overset{n,b}{\sim}$. Namely for each equivalence class $[\alpha]_{n,w}$ there is a *leaf* $L \in \mathcal{L}_w^n$, given as the hyperbolically convex hull of $[\alpha]_{n,w}$. A white/black n -gap is the closure of one component of $S_w^2 \setminus \bigcup \mathcal{L}_w^n$, or of $S_b^2 \setminus \bigcup \mathcal{L}_b^n$ respectively. The *set of white/black n -gaps* is denoted by $\mathbf{G}_w^n, \mathbf{G}_b^n$ (Section 5.2).
- The *Carathéodory semi-conjugacy* is denoted by σ (see Section 1.2). The equivalence relation induced by σ is denoted by \approx (1.3). The semi-conjugacies, equivalence relations of the white, black polynomials P_w, P_b are denoted by σ_w, σ_b and $\overset{w}{\approx}, \overset{b}{\approx}$. Identifying additionally closures of bounded Fatou components yields the equivalence relations $\overset{\mathcal{F},w}{\approx}, \overset{\mathcal{F},b}{\approx}$ (see (8.1), (8.2)).

2. THURSTON MAPS AND TILES

2.1. Expanding Thurston maps.

Definition 2.1. A *Thurston map* is an orientation-preserving, postcritically finite, branched cover of the sphere $f: S^2 \rightarrow S^2$. This means that locally f can be written as $z \mapsto z^q$, $q \geq 1$ (after suitable local, orientation preserving, homeomorphic changes

of coordinates in domain and range). More precisely for each point $v \in S^2$ there exists a $q \in \mathbb{N}$, (open) neighborhoods V, W of $v, w = f(v)$ and orientation preserving homeomorphisms $\varphi: V \rightarrow \mathbb{D}, \psi: W \rightarrow \mathbb{D}$ with $\varphi(v) = 0, \psi(w) = 0$ satisfying

$$\psi \circ f \circ \varphi^{-1}(z) = z^q,$$

for all $x \in \mathbb{D}$. The integer $q = \deg_f(v) \geq 1$ is called the *local degree* of the map at v . A point c at which the local degree $\deg_f(c) \geq 2$ is called a *critical point*. The set of all critical points is denoted by $\text{crit} = \text{crit}(f)$. *Postcritically finiteness* means that the *set of postcritical points*

$$\text{post} = \text{post}(f) := \bigcup_{j \geq 1} f^j(\text{crit})$$

is finite.

Fix a Jordan curve $\mathcal{C} \supset \text{post}$. The Thurston map is called *expanding* if

$$\text{mesh } f^{-n}(\mathcal{C}) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Here $\text{mesh } f^{-n}(\mathcal{C})$ is the maximal diameter of a component of $S^2 \setminus f^{-n}(\mathcal{C})$. This is independent of the chosen curve \mathcal{C} (see [BM, Section 1]). It is equivalent to the notion of expansion by Haïssinsky-Pilgrim [HP09] (see [Mey, Lemma 9.2]).

If f is a rational map it is expanding if and only if its Julia set is the whole sphere.

We will however not assume that f is a rational map; in fact it should be emphasized that we make no assumption on the smoothness of f .

In particular we allow expanding Thurston maps that have *periodic critical points*. An example of such a map is obtained as follows. Consider a postcritically finite rational map g whose Julia set is a Sierpiński carpet (the closure of each Fatou component is a Jordan domain, closures of distinct Fatou components are disjoint). Identify the closure of each Fatou component, the map descends to this quotient. The quotient map is an expanding Thurston map with periodic critical points.

The map

$$F = f^n$$

will always denote the iterate from Theorem 1.1. Note that $\text{post}(F) = \text{post}(f)$. Throughout this paper we denote by

$$d := \deg F = (\deg f)^n, \quad k := \# \text{post}.$$

2.2. Tiles and edges. Fix a Jordan curve $\mathcal{C} \supset \text{post}$ (this was the first step to construct the Peano curve γ) and give it an orientation. In [BM, Section 14] it was shown that we can choose \mathcal{C} to be F -invariant ($F(\mathcal{C}) \subset \mathcal{C}$), but we do not assume this here. We *color* the components of $S^2 \setminus \mathcal{C}$ *white* and *black*, such that \mathcal{C} is positively oriented as the boundary of the white component. The closures of the white/black components of $S^2 \setminus \mathcal{C}$ are called the white/black 0-tiles, denoted by X_w^0, X_b^0 . Similarly the closure of each component of $S^2 \setminus F^{-n}(\mathcal{C})$ is called an *n-tile*. It is colored white if it is mapped to X_w^0 by F^n , black if mapped to X_b^0 by F^n . The set of all n -tiles is denoted by \mathbf{X}^n . The restricted map

$$(2.1) \quad F^n: X^n \rightarrow X_{w,b} \text{ is a homeomorphism}$$

for each $X^n \in \mathbf{X}^n$, see [BM, Prop 6.1]. In particular each n -tile is a closed Jordan domain.

The postcritical points divide the curve \mathcal{C} into k (closed) arcs, called the *0-edges*. A preimage of a 0-edge by a homeomorphism as in (2.1) is called an *n-edge*. Thus the boundary of each *n-tile* consists of k *n-edges*. The set of all *n-edges* is denoted by \mathbf{E}^n .

The preimage of a postcritical point by the homeomorphism from (2.1) is called an *n-vertex*. Each *n-tile* contains k *n-vertices* in its boundary. Sometimes a postcritical point is also called a *0-vertex* for convenience. The set of all *n-vertices* is denoted by \mathbf{V}^n . Note that

$$\text{post} = \mathbf{V}^0 \subset \mathbf{V}^1 \subset \dots$$

Expansion implies that the union of the sets \mathbf{V}^n is dense.

Recall that *n-tiles* are colored white and black. The *n-tiles* tile the sphere S^2 in a *checkerboard fashion*. This means that two *n-tiles* which share an *n-edge* are colored differently. Put differently, at each *n-vertex* v an even number of *n-tiles* intersect, their colors alternate around v .

2.3. The visual metric. In [BM, Section 15] it was shown that S^2 can be equipped with a *visual metric*, denoted by $|x-y|_S$, with respect to which F is a *local similarity*.

Theorem 2.2 ([BM]). *There is a constant $\Lambda > 1$ such that the following holds. For every $x \in S^2$ there is a neighborhood U_x of x such that*

$$\frac{|F(x) - F(y)|_S}{|x - y|_S} = \Lambda \quad \text{for all } y \in U_x.$$

Distances with respect to the visual metric can easily be estimated in *combinatorial terms*. Namely let

$$(2.2) \quad m(x, y) := \min\{n \in \mathbb{N} \mid \text{there exists disjoint } n\text{-tiles } X^n \ni x, Y^n \ni y\},$$

for all distinct $x, y \in S^2$; we set $m(x, x) = \infty$ (for all $x \in S^2$). Then (see [BM, Section 8])

$$(2.3) \quad |x - y|_S \asymp \Lambda^{-m(x, y)},$$

for all $x, y \in S^2$. Here we set $\Lambda^{-\infty} = 0$. The constant $C(\asymp) = C(\mathcal{C})$ is independent of x, y .

3. THE INVARIANT PEANO CURVE AND ITS APPROXIMATIONS

The paper present is a direct continuation of [Mey]. In particular we do not only use the main result (i.e., Theorem 1.1), but in fact the whole construction of the invariant Peano curve γ . We outline the construction here. Results of [Mey] will be used freely.

3.1. The approximations γ^n . The curve γ is constructed as the limit of curves γ^n , called the *n-th approximation* (of the Peano curve). The approximations satisfy the following.

- As a set $\gamma^0 = \mathcal{C}$, more precisely

$$\gamma^0: S^1 \rightarrow \mathcal{C}$$

is a homeomorphism.

- The γ^n cover all *n-edges*. This means when γ^n is viewed as a set it holds

$$\gamma^n = \bigcup \mathbf{E}^n.$$

- A point $\alpha^n \in S^1$ such that $\gamma^n(\alpha^n) \in \mathbf{V}^n$ is called an n -angle. Each n -angle is *rational* (here we identify S^1 with \mathbb{R}/\mathbb{Z}). The *set of all n -angles*

$$\mathbf{A}^n := (\gamma^n)^{-1}(\mathbf{V}^n)$$

is a finite set.

- For $m \geq n$ it holds that $\gamma^m = \gamma^n$ on \mathbf{A}^n ,

$$\gamma^n|_{\mathbf{A}^n} = \gamma^m|_{\mathbf{A}^n} = \gamma|_{\mathbf{A}^n}.$$

Thus (recall that $\mathbf{V}^0 \subset \mathbf{V}^1 \subset \dots$)

$$\mathbf{A}^0 \subset \mathbf{A}^1 \subset \dots$$

- The n -angles divide the circle S^1 into (closed) n -arcs. Each n -arc is mapped by γ^n homeomorphically to an n -edge. Conversely for each n -edge E^n there is a unique n -arc $a^n \subset S^1$, such that $\gamma^n(a^n) = E^n$.
- Each n -arc is mapped homeomorphically to an $(n-1)$ -arc by $z^d: S^1 \rightarrow S^1$. More precisely, we have the following commutative diagram:

$$(3.1) \quad \begin{array}{ccc} \mathbf{A}^{n+1} \subset \mathbb{R}/\mathbb{Z} & \xrightarrow{\mu} & \mathbf{A}^n \subset \mathbb{R}/\mathbb{Z} \\ \gamma^{n+1} \downarrow & & \downarrow \gamma^n \\ \mathbf{V}^{n+1} \subset S^2 & \xrightarrow{F} & \mathbf{V}^n \subset S^2. \end{array}$$

See [Mey, Lemma 4.5 (2)], as well as Remark 3.1. Thus

$$\text{diam } a^n \lesssim d^{-n}$$

for each n -arc (with some constant $C(\lesssim)$).

- The curve γ^n touches itself, but does not intersect itself. This means for any $\epsilon > 0$ there is a Jordan curve $\gamma_\epsilon^n: S^1 \rightarrow S^2$ such that

$$\|\gamma^n - \gamma_\epsilon^n\|_\infty < \epsilon.$$

- The curves γ^n converge uniformly to γ . More precisely

$$\|\gamma^n - \gamma\|_\infty \lesssim \Lambda^{-n}.$$

Here the supremums norm (as well as the constant $\Lambda > 1$) is taken with respect to the metric from Theorem 2.2.

- Each n -edge E^n is contained in the boundary of a (unique) white n -tile X_w^n . We orient ∂X_w^n mathematically positively, this induces an orientation on E^n . In other words, each 0-edge E^0 inherits an orientation from the orientation of \mathcal{C} . Then we use the homeomorphism $F^n: X_w^n \rightarrow X_w^0$ to pull back the orientation of 0-edges to each n -edge $E^n \subset X_w^n$. Note that F maps positively oriented n -edges to positively oriented $(n-1)$ -edges by definition.

Alternatively we can use the maps γ^n to define an orientation on the n -edges. This means $\gamma^n(a^n) = E^n$ for a (unique) n -arc $a^n \subset S^1$. Since γ^n is a homeomorphism on a^n , the orientation of a^n induces an orientation on E^n . The following holds for the approximations γ^n :

(\star) The two orientations on E^n described above agree.

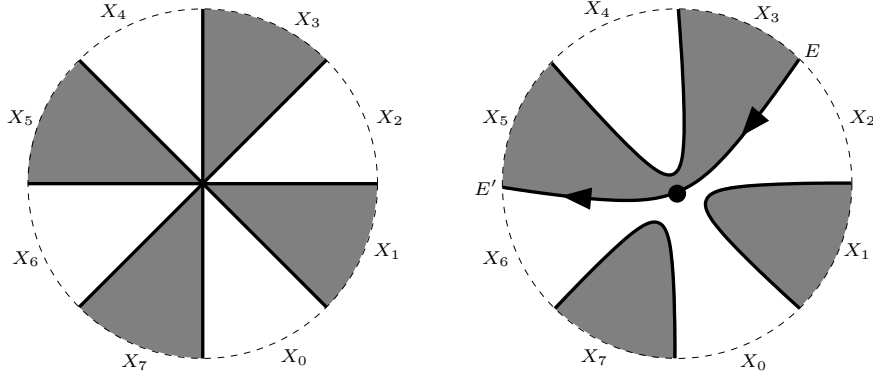


FIGURE 1. Connection at a vertex.

Remark 3.1. Note that the approximations $\tilde{\gamma}^n: \mathbb{R}/\mathbb{Z} \rightarrow S^2$ parametrized as in [Mey, Section 4.2] converge to $\tilde{\gamma}: \mathbb{R}/\mathbb{Z} \rightarrow S^2$, which semi-conjugates F to $\tilde{\mu}(t) = dt + \theta_0(\text{mod } 1)$. The approximations

$$\gamma^n(t) := \tilde{\gamma}^n(t - \theta_0/(d-1))$$

converge to

$$\gamma(t) = \tilde{\gamma}(t - \theta_0/(d-1)),$$

which is the desired Peano curve, i.e., semi-conjugates F to $\mu(t) = dt(\text{mod } 1)$, see [Mey, Lemma 4.1] and the subsequent remark. Let $\{\tilde{\alpha}_j^n\} \subset S^1$ be the points from [Mey, Section 4.2] (that are mapped by $\tilde{\gamma}^n$ to n -vertices). Then the points

$$\alpha_j^n := \tilde{\alpha}_j^n + \theta_0/(d-1)$$

are the n -angles.

3.2. Pseudo-isotopies. The approximations γ^n described in the last section were constructed as follows. A *pseudo-isotopy* $H: S^2 \times [0, 1]$ is a homotopy, that is an isotopy on $[0, 1)$ (i.e., it ceases to be a homeomorphism only at $t = 1$); if H is constant on a set A it is an *isotopy rel. A*. In [Mey] a pseudo-isotopy H^0 rel. $\text{post} = \mathbf{V}^0$ is constructed that deforms γ^0 to γ^1 , where γ^0, γ^1 are as in the last section. It is possible to *lift* H^0 by F^n to H^n . This is a pseudo-isotopy rel. \mathbf{V}^n and deforms γ^n to γ^{n+1} .

3.3. Connections. The first approximation γ^1 (more precisely the pseudo-isotopy H^0) was constructed as follows. At each 1-vertex v several white 1-tiles intersect. We declare which white 1-tiles are *connected* at v ; see Figure 1 for an illustration. Connections (of white 1-tiles at v) are *non-crossing*. Furthermore the resulting *white tile graph* forms a *spanning tree*. “Following the outline” of this tree yields the first approximation γ^1 . There is one more ingredient: the resulting curve γ^1 has to be “in the right homotopy class”. This means there has to be a pseudo-isotopy H^0 rel. post as in Section 3.2 that deforms γ^0 to γ^1 .

Formally let X_0, \dots, X_{2n-1} be the 1-tiles intersecting in a 1-vertex v , ordered mathematically positively around v . The white 1-tiles have even index, the black ones odd index. We consider a decomposition $\pi_w = \pi_w(v)$ of $\{0, 2, \dots, 2n-2\}$ (i.e., of indices corresponding to white 1-tiles around v); and a decomposition $\pi_b = \pi_b(v)$

of $\{1, 3, \dots, 2n - 1\}$ (i.e., indices corresponding to black 1-tiles around v). They satisfy the following:

- They are *decompositions*. This means $\pi_w = \{b_1, \dots, b_N\}$, where each block b_i is a subset of $\{0, 2, \dots, 2n - 2\}$, $b_i \cap b_j = \emptyset$ ($i \neq j$), and $\bigcup b_i = \{0, 2, \dots, 2n - 2\}$. Similarly for π_b .
- The decompositions π_w, π_b are *non-crossing*. This means the following. Two distinct blocks $b_i, b_j \in \pi_w$ are *crossing* if there are numbers $a, c \in b_i$, $b, d \in b_j$ and

$$a < b < c < d.$$

Each partition π_w, π_b does not contain any (pair of) crossing blocks.

- The partitions π_w, π_b are *complementary*. This means the following. Given π_w , the partition π_b is the unique, biggest partition (of $\{1, 3, \dots, 2n - 1\}$) such that $\pi_w \cup \pi_b$ is a non-crossing partition of $\{0, 1, \dots, 2n - 1\}$.

A partition $\pi_w \cup \pi_b$ as above is called a *complementary non-crossing partition*, or *cnc-partition*. A *connection* (of 1-tiles) assigns to each 1-vertex a cnc-partition as above. Two 1-tiles $X_i, X_j \ni v$ are said to be *connected at v* if the indices i, j are contained in the *same* block of $\pi_w \cup \pi_b$. Note that tiles of different color are never connected. The 1-tile X_i is *incident* to the block $b \ni i$ of $\pi_w \cup \pi_b$. In the example illustrated in Figure 1 we have $\pi_w = \{\{0, 2, 6\}, \{4\}\}$, $\pi_b = \{\{1\}, \{3, 5\}, \{7\}\}$.

Remark 3.2. In the construction of the “initial pseudo-isotopy” H^0 we need additional data to the one described above. Namely in the case when v is a postcritical point, we need to “say where v is in the connection”. This will be of no importance in the paper present.

3.4. Succeeding edges. The connection of 1-tiles from Section 3.3 can be used to define which 1-edges are *succeeding* at some 1-vertex v . Indeed this is the main purpose of the connection. Figure 1 serves again as an illustration.

Let the connection at a 1-vertex v be given by $\pi_w(v) \cup \pi_b(v)$. Two indices $i, j \in b \in \pi_w(v) \cup \pi_b(v)$ are called *succeeding* (in b), if b does not contain any index $i + 1, i + 2, \dots, j - 1$. Indices are taken mod $2n$ here, where $2n$ is the number of 1-tiles containing v .

Consider two positively oriented (as boundaries of 1-tiles) 1-edges E, E' ; where E has terminal point v , E' has initial point v . E, E' are *succeeding at v* if $E \subset X_i$, $E' \subset X_j$ and i, j are succeeding indices of a block $b \in \pi_w(v)$ (thus X_i, X_j are *white* 1-tiles). The first approximation γ^1 (viewed as an Eulerian circuit in $\bigcup \mathbf{E}^1$) may be given by the connection,

the 1-edges E, E' are *succeeding* in γ^1 if and only if
they are succeeding with respect to the connection.

3.5. Connection of n -tiles. The connection of 1-tiles can be *lifted* to a connection of n -tiles (see [Mey, Section 8]). Thus at each n -vertex v a cnc-partition $\pi_w^n(v) \cup \pi_b^n(v)$ (as in Section 3.3) is defined. Succeeding n -edges are defined as in Section 3.4. As before

the n -edges E, E' are *succeeding* in γ^n if and only if
they are succeeding with respect to the connection of n -tiles.

3.6. The connection graph. The connection of n -tiles can be used to defined the n -th white connection graph. It is constructed as follows. For each white n -tile X there is a vertex $c(X)$ (thought of as the *center* of the n -tile X); for each n -vertex v and block $b \in \pi_w^n(v)$ there is a vertex $c(v, b)$. The vertex $c(X)$ is connected to $c(v, b)$ if (and only if) X is incident to b at v . The connection of n -tiles satisfies the following.

The n -th white connection graph is a tree.

4. EQUIVALENCE RELATIONS

4.1. The lattice of equivalence relations. Equivalence relations on a set S can be *partially ordered* in a natural way, namely \sim^b is *bigger* than \sim^a ($\sim^b \geq \sim^a$) if

$$(4.1) \quad s \sim^a t \Rightarrow s \sim^b t,$$

for all $s, t \in S$. Equivalently, each equivalence class of \sim^a is a subset of some equivalence class of \sim^b .

The set of all equivalence relations on S forms a *lattice*, when equipped with this partial ordering. This means that *join* and *meet* are well defined. Recall that the join $\sim^a \vee \sim^b$ is the smallest equivalence relation bigger than \sim^a and \sim^b . If $\sim^\vee := \sim^a \vee \sim^b$, then

$$\begin{aligned} & s \sim^\vee t \text{ if and only if} \\ & \text{there are } s_1, \dots, s_N \in S \text{ such that} \\ & s = s_1 \sim^a s_2 \sim^b s_3 \dots s_{N-2} \sim^a s_{N-1} \sim^b s_N = t, \end{aligned}$$

for all $s, t \in S$.

The meet $\sim^a \wedge \sim^b$ is the biggest equivalence relation smaller than \sim^a and \sim^b . If $\sim^\wedge := \sim^a \wedge \sim^b$, then

$$(4.2) \quad s \sim^\wedge t \text{ if and only if } s \sim^a t \text{ and } s \sim^b t.$$

for all $s, t \in S$.

4.2. Closed equivalence relations. We consider an equivalence relation \sim on a topological space S .

Definition 4.1 (Closed equivalence relation). An equivalence relation \sim on a compact metric space S is called *closed* if one of the following equivalent conditions holds.

(CE 1) $\{(s, t) \mid s \sim t\} \subset S \times S$ is closed.

(CE 2) For any two convergent sequences in S ; $s_n \rightarrow s_0, t_n \rightarrow t_0$ it holds

$$s_n \sim t_n \text{ for all } n \geq 1 \Rightarrow s_0 \sim t_0.$$

(CE 3) The projection map

$$\pi: S \rightarrow S/\sim \text{ given by } s \mapsto [s]$$

is closed.

The proof that the above conditions are equivalent is straightforward and left as an exercise.

Remark 4.2. If \sim is closed as above it follows that each equivalence class is compact. Indeed by (CE 3) the set $[s] = \pi^{-1}(\pi(s))$ is closed, thus compact (for all $s \in S$).

Remark 4.3. The set of equivalence classes $\{[s] \mid s \in S\}$ forms a *decomposition* of S (i.e., a set of disjoint subsets of S whose union is S). Conversely, each decomposition can be viewed as an equivalence relation. An equivalence relation is closed if and only if the induced decomposition is *upper semicontinuous*. Property (CE 3), together with the requirement that each equivalence class is compact, is the general definition of upper semicontinuity in any topological space.

Remark 4.4. The closedness/upper semicontinuity of \sim should be viewed as the minimal requirement that the quotient space (or *decomposition* space) S/\sim has a “reasonable” topology. For example (S is a compact metric space) \sim is closed if and only if

(CE 4) S/\sim is Hausdorff.

The necessity follows immediately from (CE 2), see [Dav86, Proposition 2.1] for the sufficiency (this is the standard reference on decomposition spaces).

Lemma 4.5 (Closure of equivalence relation). *Let \sim be an equivalence relation on a compact metric space S . Then there is a unique smallest closed equivalence relation $\hat{\sim}$ bigger than \sim . We call $\hat{\sim}$ the closure of \sim .*

Proof. Consider the family $\{\overset{\alpha}{\sim}\}_{\alpha \in I}$ of all closed equivalence relations bigger than \sim . This family is non-empty. Let $\hat{\sim}$ be their meet

$$s \hat{\sim} t := \bigwedge_{\alpha} \overset{\alpha}{\sim} \quad \Leftrightarrow \quad s \overset{\alpha}{\sim} t \text{ for all } \alpha \in I.$$

Each equivalence class of $\hat{\sim}$ is the intersection of equivalence classes of $\overset{\alpha}{\sim}$, thus compact. It remains to show that $\hat{\pi}: S \rightarrow S/\hat{\sim}$ is closed (CE 3). Let $A \subset S$ be closed. Then $\hat{\pi}(A) \subset S/\hat{\sim}$ is closed if and only if $\hat{A} := \hat{\pi}^{-1}(\hat{\pi}(A))$ is closed. Note that

$$\hat{A} = \{s \in S \mid s \hat{\sim} a \text{ for a } a \in A\} = \bigcap_{\alpha} A_{\alpha}, \text{ here}$$

$$A_{\alpha} = \{s \in S \mid s \overset{\alpha}{\sim} a \text{ for a } a \in A\} = \pi_{\alpha}^{-1}(\pi_{\alpha}(A)),$$

where $\pi_{\alpha}: S \rightarrow S/\overset{\alpha}{\sim}$. The sets A_{α} are closed since $\overset{\alpha}{\sim}$ is closed, thus \hat{A} is closed. \square

Note that $\{(s, t) \mid s \hat{\sim} t\}$ is generally *not* the closure of $\{(s, t) \mid s \sim t\}$, which may fail to be transitive.

4.3. Equivalence relation induced by γ . A surjection $h: S \rightarrow S'$ induces an equivalence relation in a natural way. Under mild assumptions the quotient S/\sim is homeomorphic to S' .

Lemma 4.6 (Equivalence relation induced by a map). *Let S, S' be compact Hausdorff spaces, and $h: S \rightarrow S'$ a continuous surjection. Define an equivalence relation on S by $s \sim t$ if and only if $h(s) = h(t)$. Then S/\sim is homeomorphic to S' . The homeomorphism is given by $[s] \mapsto h(s)$.*

Proof. This is [HY88, Theorem 3-37]. \square

The equivalence relation \sim on S^1 is now the one induced by γ ,

$$s \sim t \Leftrightarrow \gamma(s) = \gamma(t),$$

for all $s, t \in S^1$.

The previous lemma together with Theorem 1.1 now yields Theorem 1.3.

Consider now the equivalence relations $\overset{n}{\sim}$ induced by γ^n and their join $\overset{\infty}{\sim}$,

$$(4.3) \quad s \overset{n}{\sim} t \text{ if and only if } \gamma^n(s) = \gamma^n(t);$$

$$(4.4) \quad \overset{\infty}{\sim} := \bigvee \overset{n}{\sim}, \text{ meaning } s \overset{\infty}{\sim} t \text{ if and only if } s \overset{n}{\sim} t \text{ for some } n;$$

for all $s, t \in S^1$.

Theorem 4.7. *The equivalence relation \sim on S^1 induced by γ (1.1) is the closure of $\overset{\infty}{\sim}$.*

To prove this theorem we will need some preparations first. Consider two points $s, t \in S^1$ such that $\gamma(s) = \gamma(t)$. We want to show that s, t are equivalent with respect to the closure of $\overset{\infty}{\sim}$. Recall that $\gamma^n(s), \gamma^n(t)$ are contained in some n -edges by construction.

Lemma 4.8. *There is a constant N (independent of s, t , and n) such that*

$$\gamma^n(s), \gamma^n(t) \text{ can be joined by at most } N \text{ } n\text{-edges,}$$

for all $s, t \in S^1$ with $\gamma(s) = \gamma(t)$.

Proof. Recall from (2.3) that for all $x, y \in S^2$

$$|x - y|_S \asymp \Lambda^{-m},$$

where $m = m(x, y)$ is the smallest number for which there exist disjoint m -tiles $X^m \ni x, Y^m \ni y$. Fix $s, t \in S^1$ with $\gamma(s) = \gamma(t)$. It holds (see Section 3.1)

$$|\gamma^n(s) - \gamma^n(t)|_S \lesssim \Lambda^{-n},$$

with a constant $C(\lesssim)$ independent of s, t , and n . Thus, there is a constant n_0 such that the $(n - n_0)$ -tiles

$$X^{n-n_0} \ni \gamma^n(s), Y^{n-n_0} \ni \gamma^n(t) \text{ are not disjoint.}$$

We now want to cover X^{n-n_0}, Y^{n-n_0} by n -tiles. The number required may be unbounded (since we do not assume that \mathcal{C} is F -invariant). Given an n -vertex v the n -flower $W^n(v)$ around v is defined as the union of all n -tiles containing v . Then

every $(n - n_0)$ -tile can be covered by M n -flowers,

where the number M is independent of n, n_0 (see [BM, Section 7]). Clearly in any n -flower $W^n(v)$ we can connect any two n -edges $E_1, E_2 \subset W^n(v)$ by at most $2k$ n -edges in $W^n(v)$. Thus $\gamma^n(s), \gamma^n(t)$ can be connected by at most $4kM$ n -edges. \square

Proof of Theorem 4.7. Fix $s, t \in S^1$ with $\gamma(s) = \gamma(t)$. According to the last lemma let

$$E_1^n, \dots, E_N^n, \text{ with } \gamma^n(s) \in E_1^n, \gamma^n(t) \in E_N^n,$$

be a chain of n -edges for each n . We can assume that E_1^n, E_N^n are the images of n -arcs containing s, t by γ^n . By taking a subsequence we can assume that N , the number of n -edges in this chain, is the same for all n .

Let $[u_j^n, v_j^n] \subset \mathbb{R}/\mathbb{Z}$ be the n -arc that is mapped to E_j^n , $\gamma^n([u_j^n, v_j^n]) = E_j^n$. Then

$$v_j^n \overset{n}{\sim} u_{j+1}^n \text{ for } j = 1, \dots, N-1.$$

Taking subsequences, we can assume that all the sequences $(v_j^n)_{n \in \mathbb{N}}$, $(u_j^n)_{n \in \mathbb{N}}$ converge. Thus (for all $j = 1, \dots, N$)

$$\begin{aligned} \lim_n u_j^n &= \lim_n v_j^n =: v_j \text{ and} \\ \lim_n v_1^n &= s, \quad \lim_n v_N^n = t. \end{aligned}$$

Now let $\hat{\sim}$ be the closure of \sim . Then

$$s = v_1 \hat{\sim} v_2 \hat{\sim} \dots \hat{\sim} v_N = t.$$

Hence $s \hat{\sim} t$, meaning that $\hat{\sim}$ is bigger than \sim . \square

5. THE WHITE AND BLACK EQUIVALENCE RELATIONS

In view of Theorem 1.3 and Theorem 4.7, it is possible to recover the map F (up to topological conjugacy) from the equivalence relations \sim^n , i.e., the self intersections of the approximations γ^n . Here we show that \sim^n can be constructed inductively. This means one can recover F topologically from finite data.

The proper setting is the following. We “break up” the equivalence relation \sim^n into two equivalence relations $\sim^{n,w}, \sim^{n,b}$ (on S^1). They describe where γ^n touches itself “in the white component”, or “in the black component”. We have

$$\sim^n = \sim^{n,w} \vee \sim^{n,b}.$$

The equivalence relations $\sim^{n,w}, \sim^{n,b}$ can be obtained inductively from $\sim^{1,w}, \sim^{1,b}$.

Both equivalence relations will be *non-crossing*. This means the following. Given two distinct equivalence classes $[s]_{n,w}, [t]_{n,w} \subset \mathbb{R}/\mathbb{Z}$ of $\sim^{n,w}$ there are no points $a, c \in [s]_{n,w}, b, d \in [t]_{n,w}$ such that

$$(5.1) \quad 0 \leq a < b < c < d < 1.$$

Note that sometimes the term “unlinked” has been used for what we call non-crossing.

It will be convenient to represent the equivalence relations geometrically in two different ways. View $S^1 \subset S^2$ as the equator. The components (hemispheres) of $S^2 \setminus S^1$ are denoted by S_w^2, S_b^2 . The circle S^1 is positively oriented as the boundary of the *white* hemisphere S_w^2 (negatively oriented as boundary of the *black* hemisphere S_b^2). We equip S_w^2, S_b^2 with the hyperbolic metric (solely to be able to talk about *hyperbolic geodesics*).

For each equivalence class $[s]_{n,w} \subset S^1 \subset S^2$ of $\sim^{n,w}$ there is a *leaf*, which is the hyperbolically convex hull of $[s]_{n,w}$ in S_w^2 . The *set of all leaves* is the *lamination* \mathcal{L}_w^n . Similarly the lamination \mathcal{L}_b^n is defined in terms of $\sim^{n,b}$ (with leaves in S_b^2). That $\sim^{n,w}$ is non-crossing is equivalent with the fact that distinct leaves in \mathcal{L}_w^n are disjoint.

The second way to geometrically represent the equivalence relations $\sim^{n,w}, \sim^{n,b}$ is via *gaps*. A *white n-gap* is the closure of one component of $S_w^2 \setminus \bigcup \mathcal{L}_w^n$.

5.1. An example. The construction will be illustrated first for the example g from [Mey, Section 1.4]. It is a Lattès map and may be obtained as follows. Consider the equivalence relation $z \simeq \pm z + m + ni$, for all $m, n \in \mathbb{Z}$, on \mathbb{C} . The map $z \mapsto 2z$ descends to the quotient \mathbb{C}/\simeq . Then g is topologically conjugate to $2z/\simeq : \mathbb{C}/\simeq \rightarrow \mathbb{C}/\simeq$. The degree of the map is 4. For illustrative purposes it is best to lift objects to \mathbb{C} , technically speaking we work in the *orbifold covering*.

The first approximation γ^1 is illustrated to the left of Figure 2 (by the thick line). The picture also illustrates how the 4 white 1-tiles are connected (see Section 3.5) at the 3 1-vertices in the middle. The 1-angles $2/16$ and $10/16$ are both mapped by γ^1 to the point in the middle. The set $\{2/16, 10/16\}$ forms one equivalence class of $\overset{1,w}{\sim}$. The non-trivial equivalence classes (i.e., the ones which are not singletons) are

$$(5.2) \quad \text{equivalence classes of } \overset{1,w}{\sim}: \left\{ \frac{2}{16}, \frac{10}{16} \right\}, \left\{ \frac{3}{16}, \frac{7}{16} \right\}, \left\{ \frac{11}{16}, \frac{15}{16} \right\}.$$

The curve γ^1 also “touches itself in black component”. This however is not as easily seen in the orbifold covering, i.e., in Figure 2. The corresponding non-trivial equivalence classes of $\overset{1,b}{\sim}$ are

$$(5.3) \quad \text{equivalence classes of } \overset{1,b}{\sim}: \left\{ \frac{1}{16}, \frac{5}{16} \right\}, \left\{ \frac{6}{16}, \frac{14}{16} \right\}, \left\{ \frac{9}{16}, \frac{13}{16} \right\}.$$

The hyperbolic geodesic (in the disk) connecting the two points of one equivalence class of $\overset{1,w}{\sim}$ is a *leaf*. The lamination \mathcal{L}_w^1 is the set of these three leaves.

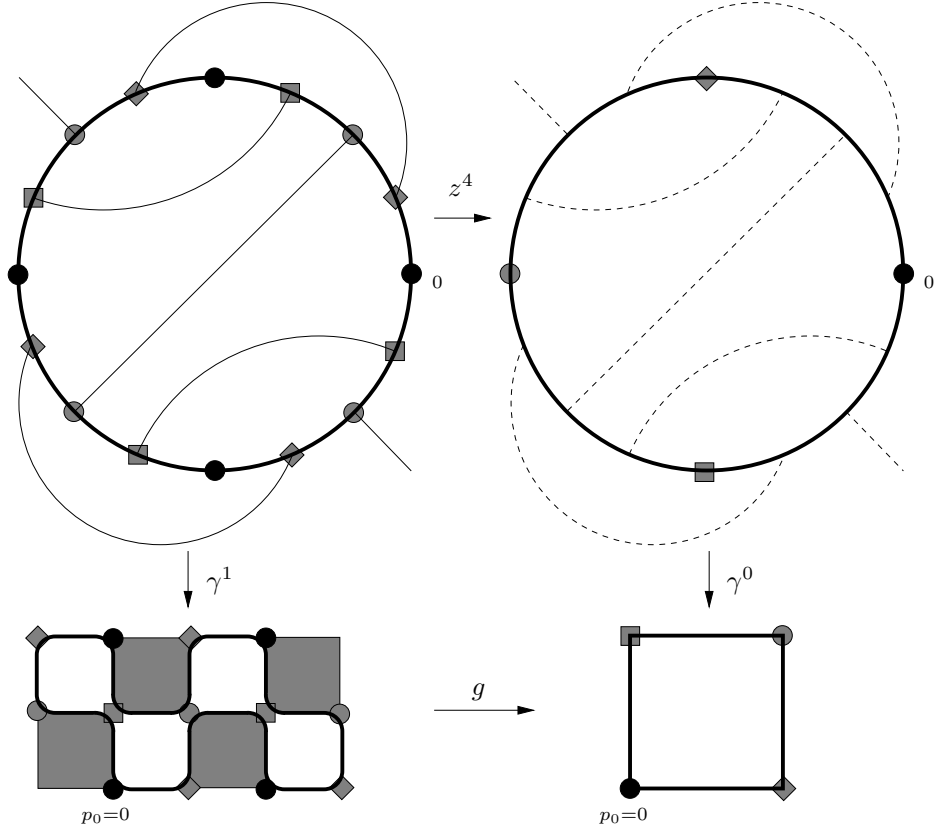
Similarly we connect the points of one equivalence class of $\overset{1,b}{\sim}$ by a hyperbolic geodesic in the outside of the circle (which we identify with the hemisphere S_b^2). The lamination \mathcal{L}_b^1 is the set of the three leaves thus obtained. The laminations $\mathcal{L}_w^1, \mathcal{L}_b^1$ are illustrated in the top left of Figure 2.

The equivalence relations $\overset{n,w}{\sim}, \overset{n,b}{\sim}$ are constructed in the same fashion. They can however be obtained from $\overset{1,w}{\sim}, \overset{1,b}{\sim}$ as follows. Consider a *white 1-gap* G , i.e., the closure of a component of $S_w^2 \setminus \bigcup \mathcal{L}_w^1$ (the interior of the circle in the top left of Figure 2). Note that each 1-gap has 4 1-arcs in its boundary, one of each type. Thus $G \cap S^1$ is mapped by $\mu(t) = 4t$ (the map z^4 when $S^1 = \mathbb{R}/\mathbb{Z}$) to the whole circle. The following holds for all $s, t \in S^1 = \mathbb{R}/\mathbb{Z}$:

$$s \overset{2,w}{\sim} t \text{ if and only if} \\ \text{there is a white 1-gap } G \ni s, t \text{ and } \mu(s) \overset{1,w}{\sim} \mu(t);$$

thus the second white equivalence relation $\overset{2,w}{\sim}$ is obtained from $\overset{1,w}{\sim}$. In the same fashion all equivalence relations $\overset{n,w}{\sim}$ can be inductively constructed from $\overset{1,w}{\sim}$; similarly all $\overset{n,b}{\sim}$ can be constructed from $\overset{1,b}{\sim}$. Thus the lists (5.2), (5.3) contain all information to recover the map g up to topological conjugacy (by Theorem 4.7 and 1.3). They are called the white and black *critical portraits* of g .

5.2. Equivalence relations, laminations, and gaps. We now define the equivalence relations $\overset{n,w}{\sim}, \overset{n,b}{\sim}$ in general. They will be defined in terms of which white/black n -tiles are connected at some n -vertex, i.e., in terms of the *connection of n -tiles* (see Section 3.5).

FIGURE 2. The laminations $\mathcal{L}^1_{w,g}, \mathcal{L}^1_{b,g}$.

Let v be an n -vertex. Consider one block $b \in \pi_w^n(v) \cup \pi_b^n(v)$ (from the cnc-partition defining the connection at v). Let X_0, \dots, X_{2m-1} be the n -tiles containing v . We call an n -edge $E \ni v$ *incident to b at v* if

$$E \subset X_i, \quad \text{where } i \in b.$$

Each n -edge is incident to exactly one block $b \in \pi_w^n(v)$ and one block $c \in \pi_b^n(v)$. Succeeding n -edges E, E' (at v) are incident to the same white/black block (see [Mey, Lemma 6.13]).

Consider an n -angle $\alpha_j^n \in S^1$ such that $\gamma^n(\alpha_j^n) = v$. It is called *incident to the block $b \in \pi_w^n(v) \cup \pi_b^n(v)$ at v* if the n -arc $a_j^n = [\alpha_j^n, \alpha_{j+1}^n]$ (as well as $a_{j-1}^n = [\alpha_{j-1}^n, \alpha_j^n]$) is mapped by γ^n to an n -edge incident to b at v . Recall that the set of n -angles $\mathbf{A}^n = \{\alpha_j^n\} \subset S^1$ is the set of all points that are mapped by γ^n to some n -vertex.

Definition 5.1 (Equivalence relations $\sim^{n,w}, \sim^{n,b}$). Let the connection at any n -vertex v be given by the cnc-partition $\pi_w^n(v) \cup \pi_b^n(v)$.

Define the equivalence relations $\overset{n,w}{\sim}, \overset{n,b}{\sim}$ on \mathbf{A}^n by

$$\begin{aligned} \alpha \overset{n,w}{\sim} \alpha' &: \Leftrightarrow \alpha, \alpha' \text{ are incident to the same block } b \in \pi_w^n(v), \\ &\text{at some } v \in \mathbf{V}^n; \\ \alpha \overset{n,b}{\sim} \alpha' &: \Leftrightarrow \alpha, \alpha' \text{ are incident to the same block } b \in \pi_b^n(v), \\ &\text{at some } v \in \mathbf{V}^n; \end{aligned}$$

for all $\alpha, \alpha' \in \mathbf{A}^n$. Thus there is a one-to-one correspondence between blocks $b \in \pi_w^n(v)$ and equivalence classes with respect to $\overset{n,w}{\sim}$; analogously between blocks $b \in \pi_b^n(v)$ and equivalence classes with respect to $\overset{n,b}{\sim}$. Note that n -angles α, α' such that $\gamma^n(\alpha) \neq \gamma^n(\alpha')$ are never equivalent.

It will be convenient to consider the equivalence relations $\overset{0,w}{\sim}, \overset{0,b}{\sim}$ on \mathbf{A}^0 as well. They are defined to be trivial, meaning that each equivalence class is a singleton.

We can view $\overset{n,w}{\sim}, \overset{n,b}{\sim}$ as equivalence relations on S^1 . Namely each $s \in S^1 \setminus \mathbf{A}^n$ is equivalent to itself and only to itself with respect to both $\overset{n,w}{\sim}$ and $\overset{n,b}{\sim}$. Let us record that $\overset{n,w}{\sim}, \overset{n,b}{\sim}$ generate the equivalence relation $\overset{n}{\sim}$, which is an immediate consequence of [Mey, Lemma 6.2],

$$(5.4) \quad \overset{n}{\sim} = \overset{n,w}{\sim} \vee \overset{n,b}{\sim}.$$

We now view $S^1 \subset S^2$ as the equator.

Definition 5.2 (Laminations $\mathcal{L}_w^n, \mathcal{L}_b^n$). Let $\alpha \in \mathbf{A}^n$ and $[\alpha]_{n,w}$ be its equivalence class with respect to $\overset{n,w}{\sim}$. A *leaf* $L = L([\alpha]_{n,w})$ is the hyperbolically convex hull of $[\alpha]_{n,w}$ in S_w^2 . If $[\alpha]_{n,w}$ corresponds to the block $b \in \pi_w^n(v)$ we sometimes write $L = L(v, b)$ to indicate “where the leaf comes from”. The set of all such leaves is the *white lamination* \mathcal{L}_w^n .

If $\#[\alpha]_{n,w} = m$ the leaf $L = L([\alpha]_{n,w})$ is an *ideal m -gon*. If $[\alpha]_{n,w}$ consists of two points the corresponding leaf L is the *hyperbolic geodesic* (in S_w^2) between these points. If $[\alpha]_{n,w}$ consists of a single point the corresponding leaf is $L = \{\alpha\}$.

The *black lamination* \mathcal{L}_b^n (containing m -gons in S_b^2) is constructed similarly from the equivalence classes of $\overset{n,b}{\sim}$.

Remark 5.3. A *lamination* is usually defined to be a closed set of disjoint *geodesics*. One obtains a lamination in this standard sense by taking the boundaries of all our leaves. We use this slightly non-standard terminology, since we do not want to make a distinction between hyperbolic geodesics and ideal m -gons.

Remark 5.4. We will formulate the following lemmas only for the *white* equivalence relations $\overset{n,w}{\sim}$, and the corresponding white laminations \mathcal{L}_w^n . In each case there is an obvious analog for *black* equivalence relations $\overset{n,b}{\sim}$, and the corresponding black laminations \mathcal{L}_b^n . More precisely one has to replace “white” by “black”, and each index “ w ” by “ b ”. Furthermore in Lemma 5.9 “cyclically” has to be replaced by “anti-cyclically”.

The angles of each equivalence class $[\alpha^n]_{n,w} \subset S^1$ inherit the cyclical ordering from S^1 .

Definition 5.5 (Succeeding). Two n -angles $\alpha^n, \tilde{\alpha}^n$ are called *succeeding* (with respect to $\overset{n,w}{\sim}$) if

$$\alpha^n \overset{n,w}{\sim} \tilde{\alpha}^n \quad \text{and} \\ \tilde{\alpha}^n \text{ is the successor to } \alpha^n \text{ in } [\alpha^n]_{n,w}.$$

This means that the open arc $(\alpha^n, \tilde{\alpha}^n) \subset S^1$ does not contain any n -angle from $[\alpha^n]_{n,w}$. The n -angle $\tilde{\alpha}^n$ is the *predecessor* to α^n .

A finite sequence $\alpha_1^n, \dots, \alpha_k^n$ is called *succeeding* (with respect to $\overset{n,w}{\sim}$) if $\alpha_j^n, \alpha_{j+1}^n$ are succeeding with respect to $\overset{n,w}{\sim}$ (for all $1 \leq j \leq k-1$). Clearly $\alpha^n \overset{n,w}{\sim} \tilde{\alpha}^n$ if and only if there is a sequence of succeeding n -angles from α^n to $\tilde{\alpha}^n$ (\mathbf{A}^n is a finite set).

Two succeeding angles correspond to a hyperbolic geodesic which forms a boundary arc of some leaf of the lamination.

The following lemma gives an alternative way to construct $\overset{n,w}{\sim}$.

Lemma 5.6. Consider two n -angles $\alpha_i^n, \alpha_j^n \in S^1$ that are mapped by γ^n to the same n -vertex v . Then

$$\alpha_i^n, \alpha_j^n \text{ are succeeding}$$

if and only if

$$\text{The } n\text{-arcs } a_i^n = [\alpha_i^n, \alpha_{i+1}^n], a_{j-1}^n = [\alpha_{j-1}^n, \alpha_j^n] \text{ are mapped by } \gamma^n \\ \text{to } n\text{-edges in the same white } n\text{-tile } X.$$

In this case the n -edge $E' = \gamma^n(a_i^n)$ succeeds $E = \gamma^n(a_{j-1}^n)$ in ∂X .

Proof. Note that both conditions imply that α_i^n, α_j^n are incident to the same block $b \in \pi_w^n(v)$.

Let X_0, \dots, X_{m-1} be the white n -tiles incident to b (at v), cyclically ordered around v . Let $E_j, E'_j \subset X_j$ be the n -edges with terminal/initial point v . Then the cyclical order of the n -edges incident to b around v is $E'_0, E_0, E'_1, E_1, \dots, E'_{m-1}, E_{m-1}$. Recall from Section 3.5 that n -edges are succeeding in γ^n (at v) if and only if they are succeeding with respect to $\pi_b^n(v) \cup \pi_w^n(v)$. Thus E_l is succeeded by E'_{l+1} in γ^n (index l is taken mod m), by definition (see Section 3.4).

The n -edges E_l, E'_{l+1} as above divide the Eulerian circuit γ^n into m chains between E'_l and E_l . Recall that γ^n is non-crossing (we can change γ^n slightly in a neighborhood of each n -vertex to obtain a Jordan curve). Thus the chain from E'_l to E_l on γ^n does not contain any other n -edge incident to b (i.e., any E_k, E'_k for $k \neq l$).

Then α_i^n, α_j^n are succeeding with respect to $\overset{n,w}{\sim}$ if and only if for $a_i^n = [\alpha_i^n, \alpha_{i+1}^n]$, $a_{j-1}^n = [\alpha_{j-1}^n, \alpha_j^n]$ it holds $\gamma^n(a_i^n) = E'_l$, $\gamma^n(a_{j-1}^n) = E_l$ (for some $l = 0, \dots, m-1$). Note that E'_l, E_l are contained in the same white n -tile by definition. \square

Note that the previous lemma remains valid if $[\alpha_i^n]_{n,w}$ consists of a single point, then of course $\alpha_i^n = \alpha_j^n$.

Recall that $\mu: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$, $\mu(t) = dt(\text{mod } 1)$. The *image* of an ideal m -gon L (in S_w^2 or S_b^2) with vertices $\alpha_{j_1}^n, \dots, \alpha_{j_m}^n$ by μ is the ideal \tilde{m} -gon \tilde{L} with vertices $\mu(\alpha_{j_1}^n), \dots, \mu(\alpha_{j_m}^n)$. The ideal \tilde{m} -gon \tilde{L} lies in the same hemisphere (in S_w^2 or S_b^2) as L . We write $\mu(L) = \tilde{L}$.

Lemma 5.7 (Properties of $\overset{n,w}{\sim}$ and \mathcal{L}_w^n). *The equivalence relations $\overset{n,w}{\sim}$ and the laminations \mathcal{L}_w^n satisfy the following.*

(\mathcal{L}^n 1) *The non-trivial equivalence classes are in \mathbb{Q} and are mapped by the iterate μ^n to a single point,*

$$[\alpha^n]_{n,w} \subset \mathbb{Q}, \quad \mu^n([\alpha^n]_{n,w}) = \{\alpha^0\} \in \mathbf{A}^0,$$

for all $\alpha^n \in \mathbf{A}^n$.

(\mathcal{L}^n 2) *The equivalence relations $\overset{n,w}{\sim}$ are non-crossing (see (5.1)). This means that the leaves in \mathcal{L}_w^n are disjoint.*

(\mathcal{L}^n 3) *Let $\{L_j\} = \mathcal{L}_w^n$, and each L_j an ideal m_j -gon. Then*

$$\sum_j m_j = kd^n, \quad \sum_j (m_j - 1) = d^n - 1.$$

(\mathcal{L}^n 4) *The lamination \mathcal{L}_w^{n+1} is mapped by μ to \mathcal{L}_w^n . This means*

$$L^{n+1} \in \mathcal{L}_w^{n+1} \Rightarrow \mu(L^{n+1}) \in \mathcal{L}_w^n,$$

equivalently

$$\mu([\alpha^{n+1}]_{n+1,w}) = [\mu(\alpha^{n+1})]_{n,w} = [\alpha^n]_{n,w},$$

for all $\alpha^{n+1} \in \mathbf{A}^{n+1}$, where $\alpha^n = \mu(\alpha^{n+1}) \in \mathbf{A}^n$.

(\mathcal{L}^n 5) *For each leaf $L^n \in \mathcal{L}_w^n$ there is a leaf $L^{n+1} \in \mathcal{L}_w^{n+1}$, such that $\mu(L^{n+1}) = L^n$. More precisely let $L_1^{n+1}, \dots, L_N^{n+1}$ be the leaves that are mapped by μ to L^n . Let $L^n = L^n([\alpha^n]_{n,w})$ be an ideal m -gon ($\#[\alpha^n]_{n,w} = m$) and each $L_j^{n+1} = L^{n+1}([\alpha_j^{n+1}]_{n,w})$ an ideal m_j -gon ($\#[\alpha_j^{n+1}]_{n,w} = m_j$). Each m_j is a multiple of m and*

$$\sum_j m_j = d m.$$

(\mathcal{L}^n 6) *Restricted to \mathbf{A}^n it holds*

$$\overset{n+1,w}{\sim} = \overset{n,w}{\sim}, \text{ equivalently}$$

$$[\alpha^n]_{n,w} = [\tilde{\alpha}^n]_{n,w} \Leftrightarrow [\alpha^n]_{n+1,w} = [\tilde{\alpha}^n]_{n+1,w},$$

for all $\alpha^n, \tilde{\alpha}^n \in \mathbf{A}^n$, $n \geq 0$. In particular

$$[\alpha^n]_{n,w} \subset [\alpha^n]_{n+1,w}, \quad \text{equivalently } L^n \subset L^{n+1},$$

where $L^n = L^n([\alpha^n]_{n,w}) \in \mathcal{L}_w^n$, $L^{n+1} = L^{n+1}([\alpha^n]_{n+1,w}) \in \mathcal{L}_w^{n+1}$. This means that

$$\overset{1,w}{\sim} \leq \overset{2,w}{\sim} \leq \dots$$

Note also that this means that distinct 0-angles $\alpha^0, \tilde{\alpha}^0 \in \mathbf{A}^0$ are not equivalent with respect to any $\overset{n,w}{\sim}$.

We now shift our attention to the *complements* of the laminations.

Definition 5.8 (Gaps). The closure of one component of $S_w^2 \setminus \bigcup \mathcal{L}_w^n$ ($S_w^2 \setminus \bigcup \mathcal{L}_w^n$) is called a white (black) gap or n -gap. The set of all white n -gaps is denoted by \mathbf{G}_w^n .

Note that leaves of the lamination \mathcal{L}_w^n correspond to n -vertices, and n -arcs (i.e., the closures of components of $S^1 \setminus \bigcup \mathbf{A}^n$) correspond to n -edges. Each white n -gap G corresponds to a white n -tile.

Let E_0, \dots, E_{k-1} be the 0-edges ordered cyclically on \mathcal{C} , meaning mathematically positively as boundary of the white 0-tile X_w^0 . Each n -edge E^n is said to be of *type* j if $F^n(E^n) = E_j$. Similarly each n -arc $a^n \subset S^1$ is of type j if $\gamma^n(a^n)$ is, i.e., if $F^n(\gamma^n(a^n)) = E_j$.

In the same fashion let p_0, \dots, p_{k-1} be the postcritical points labeled cyclically on \mathcal{C} . Each n -vertex v is of *type* j if $F^n(v) = p_j$. Note that v is also an $(n+m)$ -vertex (for each $m \geq 0$), and might be of different type as such. A leaf $L = L(v, b) \in \mathcal{L}_w^n$ is of *type* j if v is.

Lemma 5.9 (Properties of gaps). *We have the following properties.*

(G 1) *There is one white n -gap for each white n -tile,*

$$\#\mathbf{G}_w^n = d^n.$$

(G 2) *Each gap $G \in \mathbf{G}_w^n$ has k n -arcs $\subset S^1$ in its boundary, one of each type. Their types are cyclically ordered as boundary of G . Equivalently G intersects k leaves $L \in \mathcal{L}_w^n$, one of each type, cyclically ordered on ∂G .*

(G 3) *Consider two n -arcs $a^n, b^n \subset S^1$. Then*

$$a^n, b^n \text{ are in the boundary of the same gap } G \in \mathbf{G}_w^n$$

if and only if

$$\gamma^n(a^n), \gamma^n(b^n) \text{ are contained in the same white } n\text{-tile.}$$

(G 4) *The $(n+1)$ -gaps are mapped to n -gaps by μ . That means for each gap $G^{n+1} \in \mathbf{G}_w^{n+1}$ there is a gap $G^n \in \mathbf{G}_w^n$ such that*

$$\mu(G^{n+1} \cap S^1) = G^n \cap S^1.$$

Furthermore μ is injective on the interior of $G^{n+1} \cap S^1$.

(G 5) *Every gap $G^{n+1} \in \mathbf{G}_w^{n+1}$ is contained in a (unique) gap $G^n \in \mathbf{G}_w^n$,*

$$G^n \supset G^{n+1}.$$

(G 6) *There is a constant n_0 such that the following holds. Let $\alpha^n, \tilde{\alpha}^n \in \mathbf{A}^n$ be not equivalent with respect to $\overset{n,w}{\sim}$. Then for $m \geq n + n_0$ no gap $G^m \in \mathbf{G}_w^m$ contains points from both sets $[\alpha^n]_{m,w}, [\tilde{\alpha}^n]_{m,w}$.*

Proof of Lemma 5.7 and Lemma 5.9. (\mathcal{L}^n 2) Consider an equivalence class $[\alpha]_{n,w}$, where $\alpha \in \mathbf{A}^n$. Let $b \in \pi_w^n(v)$ be the block corresponding to $[\alpha]_{n,w}$ ($\tilde{\alpha} \in [\alpha]_{n,w} \Leftrightarrow \tilde{\alpha}$ incident to b at v).

Recall from Section 3.6 that the n -th white connection graph Γ^n is a tree.

Consider an n -arc $[\beta, \beta'] \subset \mathbb{R}/\mathbb{Z} = S^1$ (between two consecutive n -angles $\beta, \beta' \in \mathbf{A}^n$) in one component of $S^1 \setminus [\alpha]_{n,w}$. Let $d \in \pi_w^n(w), d' \in \pi_w^n(w')$ be the blocks associated to $[\beta]_{n,w}, [\beta']_{n,w}$. Then $c(w, d), c(w', d')$ (the vertices in the connection graph Γ^n associated to d, d') are in the same component of $\Gamma^n \setminus c(v, b)$. Indeed $\gamma^n([\beta, \beta'])$ is contained in a white n -tile incident to both d, d' .

Assume there is an equivalence class $[\alpha']_{n,w}$ ($\alpha' \in \mathbf{A}^n$) containing n -angles in distinct components of $S^1 \setminus [\alpha]_{n,w}$ (i.e., $[\alpha]_{n,w}, [\alpha']_{n,w}$ are crossing). Let $b' \in \pi_w^n(v')$ be the associated block. Then in Γ^n the vertex $c(v', b')$ connects distinct components of $\Gamma^n \setminus c(v, b)$. Thus Γ^n is not a tree, which is a contradiction.

(G 2) Fix a white n -tile X . Let E_0, \dots, E_{k-1} be the n -edges in the boundary of X (ordered mathematically positively in ∂X). Consider the n -arc $a_{j-1}^n = [\alpha_{j-1}^n, \alpha_j^n] \subset S^1$ that is mapped by γ^n to E_0 . Let $G \in \mathbf{G}_w^n$ be the gap having a_{j-1}^n in its boundary. Consider the n -arc $a_i^n = [\alpha_i^n, \alpha_{i+1}^n]$ that is the cyclical successor to a_{j-1}^n in ∂G . Note that α_i^n is the n -angle *preceding* α_j^n with respect to $\overset{n,w}{\sim}$. From Lemma 5.6 it follows that $\gamma^n(a_i^n) = E_1$. Continuing in this fashion yields the claim (note γ^n maps *exactly one* n -arc to each n -edge).

(G 3) This follows directly from the previous argument.

(G 1) From the above it is clear that there is a *bijection* between white n -tiles and white n -gaps. Thus there are exactly d^n such components.

(Lⁿ 3) The first equality follows from the fact that $\sum m_j = \#\mathbf{A}^n$, the number of n -angles.

Recall from Section 3.6 the definition of the *connection graph*. Given a white n -tile X let $G = G(X) \in \mathbf{G}_w^n$ be the corresponding gap according to (G 3), meaning that $\gamma^n(G \cap S^1) = \partial X$. From Definition 5.1 it follows that the vertices $c(X)$, $c(v, b)$ (of the n -th white connection graph) are connected by an edge if and only if the gap $G = G(X)$ has non-empty intersection with the leaf $L(v, b) \in \mathcal{L}_w^n$.

By (G 2) it thus follows that the (n -th white) connection graph has kd^n edges. On the other hand it has $\#\mathcal{L}_w^n + d^n$ vertices and is a tree. Thus it has $\#\mathcal{L}_w^n + d^n - 1$ edges, which implies that $\#\mathcal{L}_w^n = (k-1)d^n + 1$. Hence

$$\sum_j (m_j - 1) = \sum_j m_j - \#\mathcal{L}_w^n = kd^n - ((k-1)d^n + 1) = d^n - 1.$$

(Lⁿ 4) This follows from [Mey, Lemma 6.15], Lemma 5.6, as well as the fact that F maps succeeding $(n+1)$ -edges to succeeding n -edges and $(n+1)$ -tiles to n -tiles (of the same color).

(G 4) Each $(n+1)$ -arc is mapped by μ to an n -arc. The statement follows from (G 2) and (Lⁿ 4).

(Lⁿ 5) Consider a leaf $L^n = L^n(v, b) \in \mathcal{L}_w^n$, and an n -edge $E^n \ni v$ incident to $b \in \pi_w^n(v)$ at v . Let $E^{n+1} \ni v'$ be an $(n+1)$ -edge that is mapped by F to E^n , where $F(v') = v$. Then (by [Mey, Lemma 6.15] and Lemma 5.6) the leaf $L_j^{n+1} = L^{n+1}(v', b') \in \mathcal{L}_w^{n+1}$ is mapped by μ to L^n , where E^{n+1} is incident to $b' \in \pi_w^{n+1}(v')$ at v' . If L_j^{n+1} is an ideal m_j -gon the number $d_j := m_j/m$ is the number of preimages $((n+1)$ -edges) of E^n incident to b' at v' . Thus $\sum d_j = d$.

(Lⁿ 1) Each n -angle $\alpha^n \in \mathbf{A}^n$ is rational by construction (see [Mey, Sections 4.1, 4.2]). The second claim follows from (Lⁿ 4) and the fact that each equivalence class $[\alpha^0]_{0,w}$ ($\alpha^0 \in \mathbf{A}^0$) is a singleton (recall that $\gamma^0 = \mathcal{C}$ is a Jordan curve).

The proof of (G 5) will be postponed to Section 5.3, the proofs of (Lⁿ 6) and (G 6) to Section 5.4. \square

5.3. Inductive construction of $\overset{n,w}{\sim}$. We come to the main result in this section, namely that $\overset{n,w}{\sim}$ (or equivalently \mathcal{L}_w^n) can be constructed inductively. This means that $\overset{1,w}{\sim}, \overset{1,b}{\sim}$ together allows to recover the map F up to topological conjugacy.

The inductive construction of \mathcal{L}_w^{n+1} may be paraphrased as follows. The boundary $G \cap S^1$ of each gap $G \in \mathbf{G}_w^n$ is mapped by μ^n onto S^1 . Pulling the lamination \mathcal{L}_w^1 back constructs \mathcal{L}_w^{n+1} .

Recall that $\mu^n(z) = d^n t(\text{mod } 1)$ denotes the n -th iterate of $\mu: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}/\mathbb{Z}$. For any gap $G \in \mathbf{G}_w^n$, this map is surjective on $G \cap S^1$ by (G 2), a homeomorphism on each n -arc, and may fail to be injective only at endpoints of n -arcs.

Given ${}^{n,w}\sim$ we know all white n -gaps. To construct ${}^{n+1,w}\sim$ it is enough to construct all $(n+1)$ -angles that are succeeding with respect to ${}^{n+1,w}\sim$ (Definition 5.5).

Theorem 5.10 (Inductive construction of ${}^{n,w}\sim$). *For all $\alpha^{n+1}, \tilde{\alpha}^{n+1} \in \mathbf{A}^{n+1}$ it holds*

$$\alpha^{n+1}, \tilde{\alpha}^{n+1} \text{ are succeeding with respect to } {}^{n+1,w}\sim$$

if and only if

$$\begin{aligned} &\alpha^{n+1}, \tilde{\alpha}^{n+1} \text{ are contained in the same white } n\text{-gap } G^n \text{ and} \\ &\mu^n(\alpha^{n+1}), \mu^n(\tilde{\alpha}^{n+1}) \text{ are succeeding with respect to } {}^{1,w}\sim. \end{aligned}$$

Let us emphasize that $(n+1)$ -angles in disjoint n -gaps are never succeeding, though the generated equivalence relation may identify angles from different gaps.

We first need some preparation to prove Theorem 5.10. In the following lemma we consider $(n+1)$ -arcs

$$a^{n+1}, b^{n+1} \subset S^1,$$

the corresponding $(n+1)$ -edges

$$D^{n+1} = \gamma^{n+1}(a^{n+1}), E^{n+1} = \gamma^{n+1}(b^{n+1}),$$

and arcs $A^{n+1}, B^{n+1} \subset \bigcup \mathbf{E}^n$, satisfying

$$H_1^n(A^{n+1}) = D^{n+1}, H_1^n(B^{n+1}) = E^{n+1}.$$

Here H^n is the pseudo-isotopy from which γ^{n+1} was constructed (see Section 3.2).

Lemma 5.11. *In the setting as above*

(1)

$$\begin{aligned} &a^{n+1}, b^{n+1} \text{ are contained in the boundary of} \\ &\text{the same white } n\text{-gap } G^n \end{aligned}$$

if and only if

$$\begin{aligned} &A^{n+1}, B^{n+1} \text{ are contained in the boundary of} \\ &\text{the same white } n\text{-tile } X^n. \end{aligned}$$

(2)

$$D^{n+1}, E^{n+1} \text{ are contained in the same white } (n+1)\text{-tile,}$$

implies

$$A^{n+1}, B^{n+1} \text{ are contained in the same white } n\text{-tile.}$$

Proof. (1) Consider the n -arcs $a^n \supset a^{n+1}, b^n \supset b^{n+1}$, and the n -edges $D^n := \gamma^n(a^n), E^n := \gamma^n(b^n)$.

By construction of γ^{n+1} (see [Mey, Definition 3.8]) it holds $D^{n+1} \subset H_1^n(D^n)$, $E^{n+1} \subset H_1^n(E^n)$. Thus $A^{n+1} \subset D^n, B^{n+1} \subset E^n$.

From (G 3) it follows that a^{n+1}, b^{n+1} are contained in the same white n -gap G^n if and only if $D^n \supset A^{n+1}, E^n \supset B^{n+1}$ are contained in the same white n -tile X^n .

(2) Consider the white $(n+1)$ -tile $X^{n+1} \supset D^{n+1}, E^{n+1}$. Consider interiors, $U^{n+1} := \text{int } X^{n+1}$ and $U^n := (H_1^n)^{-1}(U^{n+1}) \subset S^2 \setminus \bigcup \mathbf{E}^n$. Clearly $A^{n+1}, B^{n+1} \subset \partial U^n$.

The map H_1^n is a homeomorphism on U^n , this follows from [Mey, property (Hⁿ 3)]. Thus U^n is connected, hence in the interior of a single n -tile X^n . From [Mey, property (H⁰ 5)] it follows that X^n has the same color as X^{n+1} . \square

We show that Property (G 5) follows as a corollary.

Proof of (G 5). We want to prove that every white $(n+1)$ -gap G^{n+1} is contained in a white n -gap G^n .

Consider two $(n+1)$ -arcs $a^{n+1}, b^{n+1} \subset G^{n+1} \cap S^1$. From (G 3) it follows that the $(n+1)$ -edges $D^{n+1} := \gamma^{n+1}(a^{n+1}), E^{n+1} := \gamma^{n+1}(b^{n+1})$ are contained in the same white $(n+1)$ -tile X^{n+1} . From Lemma 5.11 (2) it follows that A^{n+1}, B^{n+1} are contained in the same white n -tile X^n . Thus, by Lemma 5.11 (1), a^{n+1}, b^{n+1} are contained in the same gap $G \in \mathbf{G}_w^n$. \square

Consider two n -arcs $a^n = [\alpha^n, \tilde{\alpha}^n], b^n = [\beta^n, \tilde{\beta}^n]$ that are cyclically consecutive in ∂G^n , for a $G^n \in \mathbf{G}_w^n$. Then we call $\beta, \tilde{\alpha}^n$ *succeeding in G^n* . Note that two n -angles are succeeding with respect to $\sim^{n,w}$ if and only if they are succeeding with respect to some white n -gap.

Proof of Theorem 5.10. (\Rightarrow) Let $\alpha^{n+1}, \tilde{\alpha}^{n+1} \in \mathbf{A}^{n+1}$ be succeeding with respect to $\sim^{n+1,w}$. Then they are contained in the same white $(n+1)$ -gap G^{n+1} . From (G 5) it follows that $\alpha^{n+1}, \tilde{\alpha}^{n+1}$ are contained in the same white n -gap G^n .

By (G 4) and (G 2) it follows that the 1-angles $\mu^n(\alpha^{n+1}), \mu^n(\tilde{\alpha}^{n+1})$ are succeeding with respect to $\sim^{1,w}$.

(\Leftarrow) Let $\alpha^{n+1}, \tilde{\alpha}^{n+1} \in \mathbf{A}^{n+1}$ be contained in the same gap $G^n \in \mathbf{G}_w^n$, such that $\mu(\alpha^{n+1}), \mu(\tilde{\alpha}^{n+1})$ are succeeding with respect to $\sim^{1,w}$. Thus they are succeeding with respect to some 1-gap G^1 .

From (G 4) and (G 5) it follows that there is an $(n+1)$ -gap $G^{n+1} \ni \alpha^{n+1}, \tilde{\alpha}^{n+1}$ such that $\mu^n(G^{n+1} \cap S^1) = G^1 \cap S^1$. Combined with (G 2) it follows that $\alpha^{n+1}, \tilde{\alpha}^{n+1}$ are succeeding with respect to G^{n+1} , thus they are succeeding with respect to $\sim^{n+1,w}$. \square

5.4. (\mathcal{L}^n 6) and (G 6). Using Theorem 5.10 we can finish the proofs of Lemma 5.7 and Lemma 5.9.

Proof of (\mathcal{L}^n 6). We need to show that

$$\alpha^n \sim^{n,w} \tilde{\alpha}^n \Leftrightarrow \alpha^n \sim^{n+1,w} \tilde{\alpha}^n,$$

for all $\alpha^n, \tilde{\alpha}^n \in \mathbf{A}^n$ (recall that $\mathbf{A}^n \subset \mathbf{A}^{n+1}$).

We first show the statement for $n = 0$, which is the following.

Claim. Let $\alpha^0, \tilde{\alpha}^0 \in \mathbf{A}^0$ be distinct. Then $\alpha^0, \tilde{\alpha}^0$ are not equivalent with respect to $\sim^{1,w}$.

To prove this claim consider distinct angles $\alpha^0, \tilde{\alpha}^0 \in \mathbf{A}^0$. By definition of \mathbf{A}^0 , $\gamma^0(\alpha^0), \gamma^0(\tilde{\alpha}^0)$ are postcritical points; in fact *different postcritical points*, since γ^0 is a Jordan curve. By construction

$$\gamma^1(\alpha^0) = \gamma^0(\alpha^0) \neq \gamma^0(\tilde{\alpha}^0) = \gamma^1(\tilde{\alpha}^0).$$

Thus $\alpha^0, \tilde{\alpha}^0$ are *not* equivalent with respect to $\sim^{1,w}$, proving the claim.

After this preparation we are ready to prove the above equivalence in general.

(\Rightarrow) Let $\alpha^n, \tilde{\alpha}^n \in \mathbf{A}^n$ be succeeding with respect to $\sim^{n,w}$. Then they are contained in the same gap $G^n \in \mathbf{G}_w^n$. Furthermore by (G 2) they are mapped by μ^n to the same point $\alpha^0 = \mu^n(\alpha^n) = \mu^n(\tilde{\alpha}^n) \in \mathbf{A}^0$. Consider the elements of $[\alpha^0]_{1,w}$,

$$\alpha^0 = \alpha_0^1, \alpha_1^1, \dots, \alpha_N^1 = \alpha^0.$$

Here $\alpha_j^1, \alpha_{j+1}^1$ are succeeding with respect to $\sim^{1,w}$. Note that by the claim above each point α_j^1 , $1 \leq j \leq N-1$ is *not* a 0-angle, thus in the *interior* of some 0-arc. Hence there is exactly *one* $(n+1)$ -angle $\alpha_j^{n+1} \in G^n$, such that $\mu^n(\alpha_j^{n+1}) = \alpha_j^1$ (for $1 \leq j \leq N-1$) by (G 4). It follows from Theorem 5.10 that in the list

$$\alpha^n =: \alpha_0^{n+1}, \alpha_1^{n+1}, \dots, \alpha_{N-1}^{n+1}, \alpha_N^{n+1} := \tilde{\alpha}^n$$

the $(n+1)$ -angles $\alpha_j^{n+1}, \alpha_{j+1}^{n+1}$ are succeeding with respect to $\sim^{n+1,w}$. Thus $\alpha^n, \tilde{\alpha}^n$ are equivalent with respect to $\sim^{n+1,w}$.

(\Leftarrow) Let $\alpha^n \sim^{n+1,w} \tilde{\alpha}^n$. We want to show that $\alpha^n \sim^{n,w} \tilde{\alpha}^n$. Consider the sequence of succeeding $(n+1)$ -angles

$$\alpha^n =: \alpha_0^{n+1}, \alpha_1^{n+1}, \dots, \alpha_N^{n+1} := \tilde{\alpha}^n.$$

Let $\alpha_k^{n+1}, k \geq 1$ be the first element (after α^n) in this sequence that is an n -angle. Since each angle $\alpha^{n+1} \in \mathbf{A}^{n+1} \setminus \mathbf{A}^n$ is contained in a single gap $G^n \in \mathbf{G}_w^n$ it follows that $\alpha_0^{n+1}, \dots, \alpha_k^{n+1}$ is contained in a single gap $G^n \in \mathbf{G}_w^n$. Note that

$$\alpha_0^1 := \mu^n(\alpha_0^{n+1}), \dots, \alpha_k^1 := \mu^n(\alpha_k^{n+1})$$

is a sequence of succeeding 1-angles, and $\alpha_0^1, \alpha_k^1 \in \mathbf{A}^0$. From the claim above it follows that $\alpha_0^1 = \alpha_k^1 \in \mathbf{A}^0$, this means that $\alpha^n \sim^{n,w} \alpha_k^{n+1}$. Continuing in this fashion we construct a sequence of succeeding n -angles from α^n to $\tilde{\alpha}^n$. \square

Proof of (G 6). We prove the claim first for $n = 0$. Consider first distinct $\alpha^0, \tilde{\alpha}^0 \in \mathbf{A}^0$. They are mapped by γ^0 , hence by γ to distinct postcritical points, $p := \gamma^0(\alpha^0), \tilde{p} := \gamma^0(\tilde{\alpha}^0)$. Since F is expanding there is n_0 , such that for $m \geq n_0$ no m -tile contains both p, \tilde{p} . Thus $\alpha^0, \tilde{\alpha}^0$ are not contained in the same white m -gap by (G 3).

Consider now non-equivalent (with respect to $\sim^{n,w}$) $\alpha^n, \tilde{\alpha}^n \in \mathbf{A}^n$. Assume $\alpha^n \tilde{\alpha}^n \in G^{m+n} \in \mathbf{G}_w^{m+n}$ for some $m \geq n_0$. Then the distinct 0-angles $\alpha^0 := \mu^n(\alpha^n), \tilde{\alpha}^0 := \mu^n(\tilde{\alpha}^n)$ are contained in the same m -gap G^m , satisfying $\mu^n(G^{m+n} \cap S^1) = G^m \cap S^1$ by (G 4). This contradicts the case $n = 0$ above.

□

5.5. The critical portrait. According to Theorem 5.10 we can recover all equivalence relations $\overset{n,w}{\sim}$ from $\overset{1,w}{\sim}$ (and all $\overset{n,b}{\sim}$ from $\overset{1,b}{\sim}$). Note that the non-trivial equivalence classes of $\overset{1,w}{\sim}, \overset{1,b}{\sim}$ (i.e., the ones containing at least two points) are mapped by γ to critical points of F .

Definition 5.12. The sets $[\alpha_j^1]_{1,w} \subset \mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z} = S^1$, $j = 1, \dots, m$ satisfy the following, meaning they form a *critical portrait*.

- μ maps all points of $[\alpha_j^1]_{1,w}$ to a single point,

$$\mu([\alpha_j^1]_{1,w}) = \{\alpha_j^0\}.$$

- $\sum_j (\#[\alpha_j^1]_{1,w} - 1) = d - 1$.
- The sets $[\alpha_1^1]_{1,w}, \dots, [\alpha_m^1]_{1,w}$ are non-crossing.

The orbit

$$\mathbf{A}^0 := \bigcup \{\mu^n(\alpha_j^1) \mid j = 1, \dots, m, n \geq 1\}$$

is a finite set.

- No set $[\alpha_j^1]_{1,w}$ contains more than one point from \mathbf{A}^0 .

The equivalence relations $\overset{n,w}{\sim}$, as well as the laminations \mathcal{L}_w^n and the gaps \mathbf{G}_w^n , are defined inductively as in Theorem 5.10.

- There is a constant n_0 such that the following holds. Let $\alpha^0, \tilde{\alpha}^0 \in \mathbf{A}^0$ be distinct. Then for $m \geq n_0$ no gap $G^m \in \mathbf{G}_w^m$ contains points from both sets $[\alpha^0]_{m,w}, [\tilde{\alpha}^0]_{m,w}$.

Properties $(\mathcal{L}^n 1)$ – $(\mathcal{L}^n 6)$ as well as $(\mathbf{G} 1)$, $(\mathbf{G} 2)$, $(\mathbf{G} 4)$ – $(\mathbf{G} 6)$ are then satisfied, which we do not prove here. The equivalence classes of $\overset{1,b}{\sim}$ form a critical portrait as well. This proves Theorem 1.4.

A. Poirier [Poi93] (extending work of Bielefeld-Fisher-Hubbard [BFH92]) has shown that for every critical portrait there is a polynomial “realizing it”, see Section 8.4. The last condition above is actually much stronger than the corresponding condition for the general case in [Poi93]. Indeed we will show in Section 8.6 that it implies that the corresponding polynomial is of a special form. Namely closures of distinct bounded Fatou components are disjoint and do not contain any critical point.

5.6. The equivalence relations $\overset{w}{\sim}, \overset{b}{\sim}$. We consider the *closure of the join* of the equivalence relations $\overset{n,w}{\sim}, \overset{n,b}{\sim}$. We will show in Section 8 that there are polynomials P_w, P_b , such that $\overset{w}{\sim}, \overset{b}{\sim}$ are the equivalence relations induced by their Carathéodory semi-conjugacies.

Definition 5.13 ($\overset{w}{\sim}, \overset{b}{\sim}$). The equivalence relation $\overset{w}{\sim}$ on S^1 is defined as follows,

$$\begin{aligned} \overset{\infty,w}{\sim} &:= \bigvee \overset{n,w}{\sim}, \text{ meaning } s \overset{\infty,w}{\sim} t \text{ if and only if } s \overset{n,w}{\sim} t \text{ for some } n; \\ \overset{w}{\sim} &\text{ is defined to be the closure of } \overset{\infty,w}{\sim}. \end{aligned}$$

The equivalence relation $\overset{b}{\sim}$ is similarly defined as the closure of $\overset{\infty,b}{\sim} := \bigvee \overset{n,b}{\sim}$.

6. EQUIVALENCE CLASSES ARE FINITE

Here we show the following.

Theorem 6.1. *Let \sim be the equivalence relation induced by γ .*

- *If F has no critical periodic cycles, there is a number $N < \infty$ such that*

$$\#[s] \leq N \text{ for all } s \in S^1.$$
- *If F has critical periodic cycles, there is (at least) one finite equivalence class $[s]$, where $\gamma(s) \notin \text{post}$.*

Proof. We prove the second claim first. Consider points $x \in S^2$ and n -arcs $a^n \subset S^1$, such that $\gamma(a^n) \ni x$. We show that the number of such n -arcs is independent of n for some points x .

Recall from Section 3.1 that there is a constant $C > 0$ such that

$$(6.1) \quad \|\gamma - \gamma^n\|_\infty \leq C\Lambda^{-n}.$$

Thus only n -arcs a^n such that the n -edges $\gamma^n(a^n)$ have “small combinatorial distance” from x can satisfy $\gamma(a^n) \ni x$.

To make the previous outline precise, fix a 0-edge E^0 . In the following E^n will always be an n -edge which is mapped by F^n to E^0 ; X^n, Y^n are n -tiles intersecting in such an n -edge E^n .

Consider the set of n -tiles that can be reached in k steps from $X^n \cup Y^n$,

$$\begin{aligned} U_0^n &= U_0^n(X^n \cup Y^n) := X^n \cup Y^n, \\ U_k^n &= U_k^n(X^n \cup Y^n) := \bigcup \{Z^n \in \mathbf{X}^n \mid Z^n \cap U_{k-1}^n \neq \emptyset\}. \end{aligned}$$

In the visual metric (see Section 2.3) it holds

$$(6.2) \quad \text{diam } U_k^n \leq Ck\Lambda^{-n},$$

for some constant $C > 0$. We want to estimate the distance of points $x \in X^n \cup Y^n$, $y \in S^2 \setminus U_k^n$. Recall from (2.3) that

$$|x - y|_S \asymp \Lambda^{-m},$$

where $m = m(x, y)$ (see (2.2)). There is a constant $M < \infty$ such that x, y can be joined by a chain of at most M^{n-m} n -tiles (see [BM, Lemma 7.11]). Thus

$$k < M^{n-m} \quad \text{or} \quad -m > \frac{\log k}{\log M} - n.$$

Thus (with respect to the metric $|x - y|_S$)

$$(6.3) \quad \text{dist}(X^n \cup Y^n, S^2 \setminus U_k^n) \geq c\Lambda^{-m} \geq ck^\alpha \Lambda^{-n},$$

for constants $c > 0$, $\alpha = \log \Lambda / \log M > 0$ independent of k, n . Consider now the n -arcs in S^1 that are mapped by γ^n into U_k^n ,

$$(6.4) \quad \begin{aligned} E_k^n &= E_k^n(X^n \cup Y^n) := U_k^n \cap \bigcup \mathbf{E}^n \\ A_k^n &= A_k^n(X^n \cup Y^n) := (\gamma^n)^{-1}(E_k^n). \end{aligned}$$

Consider an $x \in X^n \cup Y^n$. Combining (6.1) with (6.3) yields that there is a k_0 independent of n , such that

$$(6.5) \quad s \notin A_{k_0}^n \Rightarrow \gamma(s) \neq x.$$

Let $\mathcal{C}' := \mathcal{C} \setminus \text{int } E^0$, this is the union of 0-arcs distinct from E^0 . Note that $S^2 \setminus \mathcal{C}'$ is simply connected and contains no postcritical point. By (6.2) we can find a set $U_{k_0}^n = U_{k_0}^n(X^n \cup Y^n) \subset S^2 \setminus \mathcal{C}'$ as above for some sufficiently large n .

Let F^{-n} be the branch of the inverse of F^n that maps $S^2 \setminus \mathcal{C}'$ to $\text{int } X^n \cup Y^n$ (and $\text{int } E^0$ to $\text{int } E^n$, where X^n, Y^n intersect in E^n). Define

$$U_{k_0}^{2n} := F^{-n}(U_{k_0}^n) \subset X^n \cup Y^n \subset U_{k_0}^n,$$

$U_{k_0}^{3n} := F^{-n}(U_{k_0}^{2n})$, and so on. The point $x_0 \in S^2$ is defined by

$$\{x_0\} = \bigcap_j U_{k_0}^{jn}.$$

Note that $F^n: U_{k_0}^{jn} \rightarrow U_{k_0}^{(j-1)n}$ is bijective, thus the number of jn -arcs in $A_{k_0}^{jn} \subset S^1$ is independent of j . It follows from (6.5) that

$$\gamma^{-1}(x_0) \subset \bigcap_j A_{k_0}^{jn},$$

where the right hand side contains at most finitely many points.

We now prove the first claim. If there are no critical cycles there is a constant $M < \infty$, such that $\deg_{F^n}(v) \leq M$ for all $v \in S^2, n$. Since $2 \deg_{F^n}(v)$ is the number of n -tiles/ n -edges containing the n -vertex v , it follows that there are only finitely many combinatorially different sets $E_{k_0}^n$ from (6.4). Let N be the maximal number of n -edges in every $E_{k_0}^n$. As above it follows that at most N points $s \in S^1$ are mapped to (any) $x \in S^2$ by γ . \square

7. γ MAPS LEBESGUE MEASURE TO MEASURE OF MAXIMAL ENTROPY

In this section we show that γ maps Lebesgue measure on S^1 to the measure of maximal entropy on S^2 , i.e., prove Theorem 1.2.

The *measure of maximal entropy* ν for $F: S^2 \rightarrow S^2$ may be constructed as the weak limit of $1/d^n \sum_{x \in F^{-n}(x_0)} \delta_x$, where $x_0 \in S^2$ is arbitrary ($F^{-n}(x_0)$ are the preimages of x_0 under F^n).

We denote Lebesgue measure on the circle S^1 by $|A|$, it is assumed here to be normalized ($|S^1| = 1$). This is the measure of maximal entropy of the map μ meaning it is the weak limit of $\frac{1}{d^n} \sum_{w \in \mu^{-n}(z_0)} \delta_w$ for any $z_0 \in S^1$.

Let $x_0 \in S^2 \setminus \text{post}$ be a point with the smallest number of preimages by γ ; $\gamma^{-1}(x_0) = \{t_1, \dots, t_N\}$ (see Theorem 6.1). Consider preimages, $\{s_1, \dots, s_{dN}\} := \mu^{-1}(\{t_1, \dots, t_N\})$, and $\{w_1, \dots, w_d\} := F^{-1}(x_0)$. By the commutativity of the diagram from Theorem 1.1, it follows that $\gamma(\{s_1, \dots, s_{dN}\}) = \{w_1, \dots, w_d\}$ and $\gamma^{-1}(\{w_1, \dots, w_d\}) = \{s_1, \dots, s_{dN}\}$. By the minimality of x_0 it follows that $\#\gamma^{-1}(w_j) \geq N$. Thus $\#\gamma^{-1}(w_j) = N$ for all j . The same argument yields that for all $w \in F^{-n}(x_0)$ there are N points in $\mu^{-n}(\{t_1, \dots, t_N\})$ that are mapped by γ to w .

Thus the (probability) measure

$$\frac{1}{Nd^n} \sum_{s \in \mu^{-n}(\{t_1, \dots, t_N\})} \delta_s \quad (\text{on } S^1)$$

is mapped by γ to

$$\frac{1}{d^n} \sum_{w \in F^{-n}(x_0)} \delta_w \quad (\text{on } S^2).$$

Clearly the first measure converges weakly to Lebesgue measure on S^1 , and the second measure converges weakly to the measure of maximal entropy ν (of $F: S^2 \rightarrow S^2$). This proves the theorem.

8. F IS A MATING

In this section we prove Theorem 1.5. This means that we show that F is obtained as a *mating* in the case when F has no periodic critical points. The construction will however be done for the general case, in preparation to prove Theorem 1.6.

Recall from Section 1.3 the construction of mating of two polynomials P_w, P_b . Assume for now that every critical point of P_w, P_b is *strictly preperiodic*. This means that the Fatou sets of P_w, P_b consist both of a single (unbounded) component, i.e., their filled Julia sets equal their Julia sets $\mathcal{K}_w = \mathcal{J}_w, \mathcal{K}_b = \mathcal{J}_b$. Let $\sigma_{w,b}: S^1 \rightarrow \mathcal{J}_{w,b}$ be the Carathéodory semi-conjugacies of $\mathcal{J}_w, \mathcal{J}_b$. Consider the equivalence relations $\overset{w}{\approx}, \overset{b}{\approx}$ obtained from σ_w, σ_b ,

$$\begin{aligned} s \overset{w}{\approx} t &: \Leftrightarrow \sigma_w(s) = \sigma_w(t) \\ s \overset{b}{\approx} t &: \Leftrightarrow \sigma_b(-s) = \sigma_b(-t), \end{aligned}$$

for all $s, t \in S^1$. Recall from (1.4) that

$$\begin{aligned} z^d / \overset{w}{\approx} &: S^1 / \overset{w}{\approx} \rightarrow S^1 / \overset{w}{\approx} \\ z^d / \overset{b}{\approx} &: S^1 / \overset{b}{\approx} \rightarrow S^1 / \overset{b}{\approx} \end{aligned}$$

are topologically conjugate to $P_w: \mathcal{J}_w \rightarrow \mathcal{J}_w$ and $P_b: \mathcal{J}_b \rightarrow \mathcal{J}_b$. Recalling the construction of the mating of P_w, P_b we obtain the following.

Lemma 8.1. *Let P_w, P_b be monic polynomials of the same degree d , where every critical point is strictly preperiodic. Consider (the equivalence relation on S^1 generated by $\overset{w}{\approx}, \overset{b}{\approx}$)*

$$\sim := \overset{w}{\approx} \vee \overset{b}{\approx}.$$

Then the topological mating of P_w, P_b may be given (is topologically conjugate) to the map

$$z^d / \sim: S^1 / \sim \rightarrow S^1 / \sim.$$

Thus Theorem 1.5 will be proved by showing that there are polynomials P_w, P_b as above such that

$$\overset{w}{\approx} = \overset{w}{\sim}, \quad \overset{b}{\approx} = \overset{b}{\sim}.$$

Here $\overset{w}{\sim}, \overset{b}{\sim}$ are the equivalence relations from Definition 5.13. We will show that the equivalence relation $\sim = \overset{w}{\approx} \vee \overset{b}{\approx}$ from Lemma 8.1 is closed. This implies that \sim is identical to the equivalence relation induced by the invariant Peano curve γ (1.1).

We now outline the general case, where F is allowed to have periodic critical points, i.e., we outline the proof of Theorem 1.6. The Carathéodory semi-conjugacies σ_w, σ_b (for monic, postcritically finite polynomials P_w, P_b) are defined as before.

Define equivalence relations on S^1 by

$$(8.1) \quad s \stackrel{\mathcal{F},w}{\approx} t : \Leftrightarrow \sigma_w(s) = \sigma_w(t) \text{ or } \sigma_w(s), \sigma_w(t) \in \text{clos } \mathcal{F}_w,$$

$$(8.2) \quad s \stackrel{\mathcal{F},b}{\approx} t : \Leftrightarrow \sigma_b(-s) = \sigma_b(-t) \text{ or } \sigma_b(-s), \sigma_b(-t) \in \text{clos } \mathcal{F}_b,$$

for all $s, t \in S^1$. Here $\mathcal{F}_w, \mathcal{F}_b$ are bounded Fatou components of P_w, P_b . We will show that there are polynomials P_w, P_b such that

$$\stackrel{\mathcal{F},w}{\approx} = \stackrel{w}{\sim}, \quad \stackrel{\mathcal{F},b}{\approx} = \stackrel{b}{\sim}.$$

Then \sim is the closure of $\stackrel{\mathcal{F},w}{\approx} \vee \stackrel{\mathcal{F},b}{\approx}$. This is the equivalence relation induced by γ as in (1.1). Furthermore we will show that

$$z^d / \sim : S^1 / \sim \rightarrow S^1 / \sim$$

is topologically conjugate to

$$P_w \hat{\Pi} P_b : \mathcal{K}_w \hat{\Pi} \mathcal{K}_b \rightarrow \mathcal{K}_w \hat{\Pi} \mathcal{K}_b,$$

see Section 1.4 for the definition. This will prove Theorem 1.6 (using Theorem 1.3).

8.1. Julia- and Fatou-type equivalence classes. The *non-trivial equivalence classes* of $\stackrel{1,w}{\sim}$, i.e., the ones that contain at least two points, are called the (white) *critical equivalence classes*. They are mapped by γ^1 (and thus by all γ^n and γ) to critical points of F . We divide the critical equivalence classes into ones of *Fatou-type* and *Julia-type* as follows.

- If $[\alpha^1]_{1,w}$ is *periodic*, i.e., if

$$\mu^n([\alpha^1]_{1,w}) \subset [\alpha^1]_{1,w},$$

for some $n \geq 1$, it is of *periodic Fatou-type*;

- if $[\alpha^1]_{1,w}$ is the *preimage of a periodic critical cycle*, i.e., if

$$\mu^n([\alpha^1]_{1,w}) \subset [\tilde{\alpha}^1]_{1,w},$$

for some $n \geq 1$, where $[\tilde{\alpha}^1]_{1,w}$ is of periodic Fatou-type; then $[\alpha^1]_{1,w}$ is of *preperiodic Fatou-type*;

- otherwise $[\alpha^1]_{1,w}$ is of *Julia-type*, i.e., the periodic cycle that $[\alpha^1]_{1,w}$ eventually lands in does not contain any point of a critical equivalence class.

Every Fatou-type equivalence class is mapped by γ^1 to a point that is eventually mapped to a critical periodic cycle of F . However if $[\alpha^1]_{1,w}$ is of Julia-type, and $c = \gamma^1([\alpha^1]_{1,w})$, then the periodic cycle that c eventually lands in may or may not be critical. This is due to the fact that the periodic critical point of F may “come from” the black polynomial.

Consider now an equivalence class $[\alpha^n]_{n,w}$ ($\alpha^n \in \mathbf{A}^n$) with respect to $\stackrel{n,w}{\sim}$. It is defined to be of *Fatou-type/Julia-type* if $\mu^{n-1}([\alpha^n]_{n,w})$ is of Fatou-type/Julia-type

(recall $(\mathcal{L}^n 4)$). We note the following (recall that $[\alpha^n]_{n,w} \subset [\alpha^n]_{n+1,w}$ from $(\mathcal{L}^n 6)$)

$$(8.3) \quad [\alpha^n]_{n,w}, [\alpha^n]_{n+1,w} \text{ are of the same type,}$$

for all $\alpha^n \in \mathbf{A}^n$.

8.2. Sizes of equivalence classes. The main result of this subsection is the following.

Proposition 8.2. *The expanding Thurston map F has critical periodic cycles if and only if there are Fatou-type equivalence classes of $\stackrel{1,w}{\sim}$ or $\stackrel{1,b}{\sim}$.*

We need some preparation. The *degree* of a critical equivalence class $[\alpha^1]_{1,w}$ is its size,

$$(8.4) \quad d([\alpha^1]_{1,w}) := \#[\alpha^1]_{1,w}.$$

The degree of other equivalence classes will be the degree of the critical class it contains.

$$d([\alpha^n]_{n,w}) := \begin{cases} \#[\alpha^1]_{1,w}, & \text{if } [\alpha^1]_{1,w} \subset [\alpha^n]_{n,w}; \\ 1, & \text{if } [\alpha^n]_{n,w} \text{ contains no critical class.} \end{cases}$$

Note that by $(\mathcal{L}^n 6)$ there can be at most one critical class contained in $[\alpha^n]_{n,w}$, thus the above is well defined.

Consider now $[\alpha^n]_{n,w}$, where $\alpha^n \in \mathbf{A}^n$. Let (recall $(\mathcal{L}^n 4)$)

$$[\alpha^{n-1}]_{n-1,w} := \mu([\alpha^n]_{n,w}), [\alpha^{n-2}]_{n-2,w} := \mu^2([\alpha^n]_{n,w}), \dots$$

Lemma 8.3 (Size of equivalence classes). *In the setting as above it holds*

$$\#[\alpha^n]_{n,w} = d([\alpha^n]_{n,w}) \cdot d([\alpha^{n-1}]_{n-1,w}) \cdot \dots \cdot d([\alpha^1]_{1,w}).$$

Proof. The statement is clear for $n = 1$. We proceed by induction. Thus we assume the statement is true for n .

Case (1). $[\alpha^{n+1}]_{n+1,w}$ contains no angle $\alpha^n \in \mathbf{A}^n$.

Then there is one white n -gap $G^n \supset [\alpha^{n+1}]_{n+1,w}$. More precisely $[\alpha^{n+1}]_{n+1,w}$ is contained in the *interior* of the n -arcs which form $G^n \cap S^1$ (by **(G 2)**). It follows that μ^n is bijective on $[\alpha^{n+1}]_{n+1,w}$. Thus

$$\#[\alpha^{n+1}]_{n+1,w} = \#\mu^n([\alpha^{n+1}]_{n+1,w}) = \#[\alpha^1]_{1,w} = d([\alpha^1]_{1,w}).$$

On the other hand $[\alpha^{n+1}]_{n+1,w}$ contains no $\alpha^n \in \mathbf{A}^n$, thus no $\alpha^1 \in \mathbf{A}^1$. Therefore $d([\alpha^{n+1}]_{n+1,w}) = 1$.

Similarly $[\alpha^n]_{n,w}$ contains no $\alpha^{n-1} \in \mathbf{A}^{n-1}$, hence $d([\alpha^n]_{n,w}) = 1$. Repeating the argument yields

$$\#[\alpha^{n+1}]_{n+1,w} = \underbrace{d([\alpha^{n+1}]_{n+1,w}) \cdot \dots \cdot d([\alpha^2]_{2,w})}_{=1} \cdot d([\alpha^1]_{1,w}).$$

Case (2). $[\alpha^{n+1}]_{n+1,w} \supset [\tilde{\alpha}^n]_{n,w}$ for some $\tilde{\alpha}^n \in \mathbf{A}^n$.

Let $m := \#[\tilde{\alpha}^n]$. Consider a white n -gap G^n that contains two succeeding (with respect to $\overset{n,w}{\sim}$) n -angles of $[\tilde{\alpha}^n]$. Note that there are m succeeding n -angles in $[\tilde{\alpha}^n]$, thus there are m such n -gaps G^n .

Let $\tilde{\alpha}^0 := \mu^n([\tilde{\alpha}^n]_{n,w}) \ni \mathbf{A}^0$, see $(\mathcal{L}^n 4)$. Let $d := \#[\tilde{\alpha}^0]_{1,w}$. Note that the lower index “1” is *not* a misprint. By Theorem 5.10 and $(\mathcal{L}^n 6)$ there are exactly $d - 1$ points in $G^n \cap S^1 \setminus [\tilde{\alpha}^n]_{n,w}$ that are mapped by μ^n to points in $[\tilde{\alpha}^0]_{1,w}$; thus in $[\alpha^{n+1}]_{n+1,w}$. The same argument applies to each of the m gaps intersecting $[\tilde{\alpha}^n]$. Therefore

$$\#[\alpha^{n+1}]_{n+1} = m + m(d - 1) = md.$$

By inductive hypothesis it holds

$$m = \#[\tilde{\alpha}^n]_{n,w} = d([\tilde{\alpha}^n]_{n,w}) \cdot d([\tilde{\alpha}^{n-1}]_{n-1,w}) \cdots d([\tilde{\alpha}^1]_{1,w}),$$

where $[\tilde{\alpha}^j]_{j,w} := \mu^{n-j}([\tilde{\alpha}^n]_{n,w})$. Note that $[\tilde{\alpha}^j]_{j,w} \subset [\alpha^{j+1}]_{j+1,w}$, thus $d([\tilde{\alpha}^j]_{j,w}) = d([\alpha^{j+1}]_{j+1,w})$ for $j \geq 1$. By the same argument $d = d([\tilde{\alpha}^0]_{1,w}) = d([\alpha^1]_{1,w})$. The claim follows. \square

Lemma 8.4. *The equivalence class $[\alpha^n]_{n,w}$ is of Julia-type if and only if $\lim_m \#[\alpha^n]_{m,w} < \infty$. In fact then there is a m_0 (independent of n and α^n) such that*

$$[\alpha^n]_{m,w} = [\alpha^n]_{n+m_0,w},$$

for all $\alpha^n \in \mathbf{A}^n$ and $m \geq n + m_0$. Furthermore

$$\#[\alpha^n]_{n,w} \leq 2^{d-1},$$

for all Julia-type equivalence classes $[\alpha^n]_{n,w}$.

Proof. It is clear that for any Fatou-type equivalence class $[\alpha^n]_{n,w}$ it holds $\lim_m \#[\alpha^n]_{m,w} = \infty$ by Lemma 8.3.

Let $[\alpha^1]_{1,w}$ be of Julia-type ($\alpha^1 \in \mathbf{A}^0$). There is an m_0 (independent of α^1), such that $\mu^k([\alpha^1]_{1,w})$ is not contained in any critical equivalence class (of $\overset{1,w}{\sim}$) for all $k \geq m_0$.

Let $[\alpha^n]_{m,w}$ be of Julia-type, $m \geq n + m_0$, $\alpha^n \in \mathbf{A}^n$. Then $d(\mu^k([\alpha^n]_{m,w})) = 1$ for all $k \geq n + m_0$. This proves the first claim, using Lemma 8.3.

To estimate the maximal size of a Julia-type equivalence class, let m_j ($j = 1, \dots, N$) be the sizes of the Julia-type equivalence classes of $\overset{1,w}{\sim}$. From Lemma 8.3 it follows that (for any Julia-type equivalence class) $\#[\alpha^n]_{n,w} \leq \prod m_j$. Maximizing this product subject to $\sum_j (m_j - 1) = d - 1$ (see $(\mathcal{L}^n 3)$) yields the second statement. \square

Proof of Proposition 8.2. Assume F has no critical periodic cycles. Then there is a constant $M < \infty$ such that $\deg_{F^n}(c) \leq M$ for all $c \in S^2$ and n . Recall that $\deg_{F^n}(c)$ is the number of white/black n -tiles attached at the n -vertex c . Let $\alpha \in S^1$ such that $\gamma^n(\alpha) = c$. Then $[\alpha]_{n,w} \leq M$, $[\alpha]_{n,b} \leq M$. Therefore $\overset{1,w}{\sim}, \overset{1,b}{\sim}$ has no Fatou-type classes by Lemma 8.4.

Assume now that F has critical periodic points. Let us assume first that $c = F(c)$ is a critical point. Then there are at least two white/black 1-tiles containing c . Thus $(\gamma^1)^{-1}(c) = [\alpha^1]_1$ contains at least two points. Let $\{\alpha^0\} := \mu([\alpha^1]_1)$. Recall from (3.1) that $F \circ \gamma^1 = \gamma^0 \circ \mu = \gamma^1 \circ \mu$ on \mathbf{A}^1 , thus $c = F \circ \gamma^1(\alpha^1) = \gamma^1(\alpha^0)$. It follows that $\alpha^0 \in [\alpha^1]_1$, or $[\alpha^0]_1 = [\alpha^1]_1$. Therefore $[\alpha^0]_{1,w}$ or $[\alpha^0]_{1,b}$ has to contain at least

two points (since $\overset{1}{\sim} = \overset{1,w}{\sim} \vee \overset{1,b}{\sim}$). Note that this equivalence class is of periodic Fatou-type.

Now assume that $F^n(c) = c$ for some $n \geq 1$. The same argument as above yields that there is $[\alpha^n]_{n,w}$ or $[\alpha^n]_{n,b}$, without loss of generality $[\alpha^n]_{n,w}$, containing at least two points, such that $\mu([\alpha^n]_{n,w}) \subset [\alpha^n]_{n,w}$. By Lemma 8.3 one of the classes $[\alpha^n]_{n,w}, [\alpha^{n-1}]_{n-1,w} := \mu([\alpha^n]_{n,w}), \dots, [\alpha^1]_{1,w} := \mu([\alpha^n]_{n-1,w})$ has to contain a critical class, which is periodic with respect to μ^n . \square

8.3. The equivalence relation \approx . The equivalence relation \approx will be the one obtained from the Carathéodory semi-conjugacy of the white polynomial. This description was given in [Poi93].

Recall from (G 5) that each white n -gap G^n is contained in (exactly) one white $(n-1)$ -gap G^{n-1} . Here and in the following we will consider sequences $(G^n)_{n \in \mathbb{N}}$ of gaps $G^n \in \mathbf{G}_w^n$ such that

$$(8.5) \quad G^1 \supset G^2 \supset \dots$$

We write

$$[(G^n)] := \bigcap G^n \cap S^1,$$

where it is always understood that the sequence $(G^n)_{n \in \mathbb{N}}$ is as in (8.5). Define for $s, t \in S^1$

$$(8.6) \quad s \overset{G}{\sim} t :\Leftrightarrow s, t \in [(G^n)].$$

Then

$$(8.7) \quad s \approx t :\Leftrightarrow \text{there are } s_1, \dots, s_N \in S^1 \text{ such that } s = s_1 \overset{G}{\sim} s_2 \overset{G}{\sim} \dots \overset{G}{\sim} s_N = t.$$

Note that $\overset{G}{\sim}$ is not an equivalence relation, but \approx is. Both $\overset{G}{\sim}, \approx$ should be properly equipped with an index “ w ”, which we suppress. The reader should be aware that there are analogously defined relations in terms of black gaps.

Let us record the following, we set $\mathbf{A}^\infty := \bigcup \mathbf{A}^n$.

Lemma 8.5 (Properties of $[(G^n)]$). *The sets $[(G^n)]$ satisfy the following.*

- (1) $\#[(G^n)] \leq k$, recall that $k = \# \text{post}(F)$.
- (2)

If $\alpha \notin \mathbf{A}^\infty$ then α is contained in a single set $[(G^n)]$;

if $\alpha \in \mathbf{A}^\infty$ then α is contained in at most two such sets.

- (3) *Let $\alpha^n, \tilde{\alpha}^n \in \mathbf{A}^n$ be not equivalent with respect to $\overset{n,w}{\sim}$. Then*

$$\alpha^n, \tilde{\alpha}^n \text{ are not in the same set } [(G^n)].$$

- (4) *Disjoint sets $[(G^n)]$ are non-crossing.*

Proof. The first property follows from (G 2). The second from the fact that each $\alpha \notin \mathbf{A}^\infty$ is contained in a *single* n -arc for each n ; each $\alpha \in \mathbf{A}^\infty$ is contained in at most two n -arcs (which can be in the same or different sets $[(G^n)]$). The third follows from (G 6). The last property follows from (\mathcal{L}^n 2). \square

Lemma 8.6 (Properties of \approx). *The equivalence relation \approx satisfies the following.*

(1) If $[\alpha^n]_{n,w}$ is of Julia-type ($\alpha^n \in \mathbf{A}^n$) then

$$[\alpha^n]_{\approx} \cap \mathbf{A}^n = [\alpha^n]_{n,w}.$$

If $[\alpha^n]_{n,w}$ is of Fatou-type ($\alpha^n \in \mathbf{A}^n$) then

$$[\alpha^n]_{\approx} \cap \mathbf{A}^\infty = \{\alpha^n\}.$$

(2) Each equivalence class of \approx is finite, in fact

$$\#[\alpha]_{\approx} \leq (k-1)2^{d-1}.$$

(3) The number of sets $[(G^n)]$ that may form a chain, meaning the number N from (8.7), is finite; more precisely

$$N \leq 2^{d-1}.$$

Proof. (1) From Lemma 8.5 (2) it follows that distinct sets $[(G^n)]$, $[(\tilde{G}^n)]$ may only intersect in a point $\alpha^n \in \mathbf{A}^\infty$. From Lemma 8.5 (3) it then follows that if $\alpha^n, \tilde{\alpha}^n \in \mathbf{A}^n$ are not equivalent with respect to $\overset{n,w}{\sim}$, then $\alpha^n \not\approx \tilde{\alpha}^n$, or

$$[\alpha^n]_{\approx} \cap \mathbf{A}^n \subset [\alpha^n]_{n,w}.$$

Consider a Julia-type equivalence class $[\alpha^n]_{n,w}$ ($\alpha^n \in \mathbf{A}^n$). Let m_0 be the constant from Lemma 8.4, thus $[\alpha^n]_{m,w} = [\alpha^n]_{n+m_0,w}$ for all $m \geq n + m_0$. Let $\beta^m, \tilde{\beta}^m \in [\alpha^n]_{m,w}$ be succeeding (with respect to $\overset{m,w}{\sim}$); there are at most 2^{d-1} such succeeding m -angles (see Lemma 8.4).

There is a gap $G^m \in \mathbf{G}_w^m$ containing $\beta^m, \tilde{\beta}^m$. Thus succeeding angles of $[\alpha^n]_{n+m_0,w}$ are equivalent with respect to \approx , meaning

$$[\alpha^n]_{n,w} \subset [\alpha^n]_{n+m_0,w} \subset [\alpha^n]_{\approx},$$

thus $[\alpha^n]_{\approx} \cap \mathbf{A}^n = [\alpha^n]_{n,w}$ follows as desired.

Now let $[\alpha^n]_{n,w}$ be of Fatou-type ($\alpha^n \in \mathbf{A}^n$). Then $\#[\alpha^n]_{m,w} \rightarrow \infty$ as $m \rightarrow \infty$ by Lemma 8.4. If $[\alpha^n]_{m,w} \subsetneq [\alpha^n]_{m+1,w}$, then succeeding angles in $[\alpha^n]_{m,w}$ are not succeeding in $[\alpha^n]_{m+1,w}$ (see Case (2) in the proof of Lemma 8.3). Thus a set $[(G^n)]$ may contain at most one point from $[\alpha^n]_{n,w}$. From Lemma 8.5 (2) and Lemma 8.5 (3) it follows that

$$[\alpha^n]_{\approx} \cap \mathbf{A}^\infty = \{\alpha^n\}.$$

(3) From the above it follows that $N = \max \#[\alpha^n]_{n,w}$, where the maximum is taken over all Julia-type classes. The claim thus follows from Lemma 8.4.

(2) Let $G^m \in \mathbf{G}_w^m$ be the gap containing succeeding m -angles $\beta^m, \tilde{\beta}^m \in [\alpha^n]_{m,w}$, as described in the proof of (1). Then $[(G^m)]$ contains at most $k-2$ points distinct from $\beta^m, \tilde{\beta}^m$ by Lemma 8.5 (1). Thus

$$\#[\alpha^n]_{\approx} \leq \#[\alpha^n]_{m,w} + (k-2)\#[\alpha^n]_{m,w} \leq (k-1)2^{d-1},$$

by Lemma 8.4. □

Proposition 8.7. *The equivalence relation \approx is closed.*

Proof. We first show the corresponding result for \mathcal{G} , i.e., the following.

Claim. Let $(s_j), (t_j) \subset S^1$ be sequences such that

$$s_j \rightarrow s, t_j \rightarrow t, \text{ and } s_j \stackrel{G}{\sim} t_j \text{ for all } j,$$

then

$$s \stackrel{G}{\sim} t.$$

If $s_j = s$ is constant the claim follows, since the set $\{t_j \stackrel{G}{\sim} s\}$ is finite (see Lemma 8.5 (1)).

Assume now that (s_j) is not constant. Without loss of generality we can assume that (s_j) is strictly increasing, the sets $[(G_j^n)] \ni s_j, t_j$ are disjoint, and $s_j \neq t_j$ for all j . Since disjoint sets $[(G_j^n)]$ are non-crossing (Lemma 8.5 (4)) it follows that (t_j) is strictly decreasing. Let $a^n(s) \subset S^1$ be an n -arc containing s . If s is contained in two n -arcs, i.e., if $s \in \mathbf{A}^n$, we choose $a^n(s)$ as the n -arc having s as the right endpoint. Similarly let $a^n(t) \subset S^1$ be an n -arc containing t . If $t \in \mathbf{A}^n$, let $a^n(t)$ be the n -arc with t as the left endpoint.

For each n the points s_j, t_j are in the interiors of $a^n(s), a^n(t)$ respectively for sufficiently large j . Since $s_j \stackrel{G}{\sim} t_j$ it follows that $a^n(s), a^n(t)$ are contained in the same gap $G^n \in \mathbf{G}_w^n$. Thus $s, t \in [(G^n)]$ proving the claim.

Consider now sequences $(s^n), (t^n) \subset S^1$, where $s^n \rightarrow s, t^n \rightarrow t$, such that $s^n \approx t^n$ for all n . Thus by Lemma 8.6 (3) there are $s_j^n \in S^1$ such that

$$s^n = s_1^n \stackrel{G}{\sim} s_2^n \stackrel{G}{\sim} \dots \stackrel{G}{\sim} s_N^n = t^n.$$

By taking subsequences we can assume that $s_j^n \rightarrow s_j$ as $n \rightarrow \infty$, for all j . From the previous claim it follows that $s_j \stackrel{G}{\sim} s_{j+1}$ (for $j = 1, \dots, N-1$). Thus

$$s = s_1 \stackrel{G}{\sim} \dots \stackrel{G}{\sim} s_N = t,$$

meaning $s \approx t$. □

8.4. The white polynomial. A. Poirier [Poi93], extending work of Bielefeld-Fisher-Hubbard [BFH92], has shown that postcritically finite polynomials admit a combinatorial classification in terms of external rays. The result is paraphrased here, not in full generality, but only in the relevant case at hand.

Poirier's Theorem ([Poi93], [BFH92]). *Let the sets $[\alpha_j^1]_{1,w} \subset \mathbb{Q}/\mathbb{Z} \subset \mathbb{R}/\mathbb{Z} = S^1$, $j = 1, \dots, m$ form a critical portrait as in Definition 5.12. Then there is a unique monic, centered, postcritically finite polynomial P_w such that*

(Poi 1) \approx is the equivalence relation induced by the Carathéodory semi-conjugacy, meaning that

$$s \approx t \Leftrightarrow \sigma(s) = \sigma(t);$$

where $\sigma: S^1 \rightarrow \partial\mathcal{K} = \mathcal{J}$ is the extension of the Riemann map $\psi: \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}} \rightarrow \widehat{\mathbb{C}} \setminus \mathcal{K}$ (normalized by $\psi(\infty) = \infty, \psi'(\infty) > 0$), \mathcal{K} is the filled Julia set, and \mathcal{J} the Julia set of P_w . This is [Poi93, Theorem I 3.9] and [Poi93, Proposition II 3.6]. In particular each set $\sigma([\alpha]_{\approx})$ is a single point.

(Poi 2) If $[\alpha^n]_{n,w}$ is of Fatou-type, then all points in $\sigma([\alpha^n]_{n,w})$ are in the boundary of the same bounded Fatou component of P_w . Distinct sets $[\alpha^n]_{n,w}, [\tilde{\alpha}^n]_{n,w}$ are in the boundaries of distinct bounded Fatou components. This is [Poi93, Proposition II 4.7]. Furthermore for each bounded Fatou component \mathcal{F} there is a $\alpha^n \in \mathbf{A}^n$ such that $\sigma(\alpha^n) \in \text{clos } \mathcal{F}$, where $[\alpha^n]_{n,w}$ is of Fatou-type.

Since $P_w^{-1}(\sigma(\mathbf{A}^n)) = \sigma(\mathbf{A}^{n+1})$ it follows that (for every Fatou-type class $[\alpha^n]_{n,w}$)

$$\sigma([\alpha^n]_{\infty,w}) \text{ is dense}$$

in the boundary of the (bounded) Fatou component \mathcal{F} satisfying $\sigma([\alpha^n]_{n,w}) \subset \partial \mathcal{F}$.

(Poi 3) For each gap $G^n \in \mathbf{G}_w^n$ the set

$\sigma(G^n)$ is a connected subset of the Julia set of P_w .

The images of two disjoint gaps $G^n, \tilde{G}^n \in \mathbf{G}_w^n$ are disjoint:

$$G^n \cap \tilde{G}^n = \emptyset \quad \Rightarrow \quad \sigma(G^n \cap S^1) \cap \sigma(\tilde{G}^n \cap S^1) = \emptyset;$$

see [Poi93, Chapter II.2].

The *white polynomial* P_w is the one obtained from Poirier's Theorem (from the sets $[\alpha^1]_{1,w}, \alpha^1, \in \mathbf{A}^1$).

8.5. Outline of the proof of Poirier's Theorem. Poirier definition of the critical portrait of a postcritically finite polynomial is slightly different from ours. This is due to the fact that he describes general such polynomials, not just ones with “separated Fatou set” (see Proposition 8.8) as considered here. For the convenience of the reader we give a very brief outline of the proof of the main result from Poirier's Theorem, namely the existence of the polynomial P_w .

Consider a topological polynomial, i.e., a Thurston map $P: S^2 \rightarrow S^2$ such that $P^{-1}(\infty) = \infty$. It is well known that P is “Thurston equivalent” to a polynomial if and only if it has no “Levy cycle” (Theorem 5.4 and Theorem 5.5 in [BFH92]). A Levy cycle is a Jordan curve $\Gamma \subset S^2 \setminus \text{post}(P)$ such that

- each component of $S^2 \setminus \Gamma$ contains at least two postcritical points and
- some component Γ' of $P^{-j}(\Gamma)$ is isotopic rel. $\text{post}(P)$ to Γ for some j ; and the map

$$P^j: \Gamma' \rightarrow \Gamma \text{ is of degree 1.}$$

To prove Poirier's Theorem (in our special case) one constructs a (postcritically finite) topological polynomial from the critical portrait. For each $\alpha^1 \in \mathbf{A}^1$ there is an “extended external ray” $R(\alpha^1)$ that is mapped by P to $R(\mu(\alpha^1))$. The extended external rays associated to the angles of one equivalence class $[\alpha^1]_{1,w}$ intersect in a point. Assume there is a Levy cycle Γ (i.e., P is not equivalent to a polynomial). We can choose Γ in such a way that Γ intersects no preperiodic extended external ray (Lemma 8.7 in [BFH92]). From the last property of the critical portrait (Definition 5.12) it follows that two postcritical points are separated by some preperiodic extended external rays. Thus Γ contains at most one postcritical point in its interior, giving a contradiction.

8.6. **The Fatou set of P_w .** Here we show that the Fatou set of P_w is “separated”.

Proposition 8.8. *The Fatou set of P_w has the following property.*

- *The closures of two distinct bounded components $\mathcal{F}_1, \mathcal{F}_2$ of the Fatou set of P_w are disjoint,*

$$\text{clos } \mathcal{F}_1 \cap \text{clos } \mathcal{F}_2 = \emptyset.$$

- *No bounded Fatou component of P_w contains a point $\sigma([\alpha^n]_{n,w})$, where $[\alpha^n]_{n,w}$ is of Julia-type, in its boundary (recall from (Poi 1) and Lemma 8.6 (1) that σ maps $[\alpha^n]_{n,w}$ to a single point).*

We need some preparation to prove this proposition. The key is an explicit description of the set of angles in S^1 that are mapped by σ to the boundary of a given bounded Fatou component/critical point (or more generally n -vertex).

Fix an equivalence class $[\alpha^n]_{m,w}$ ($\alpha^n \in \mathbf{A}^n$). Let $G_1^m, \dots, G_{N_m}^m \in \mathbf{G}_w^m$ be the gaps intersecting $[\alpha^n]_{m,w}$. Here $N_m = \#[\alpha^n]_{m,w}$.

Lemma 8.9. *In the setting as above*

$$\bigcup_j G_j^{m+n_0} \cap S^1 \text{ is compactly contained in } \bigcup_j G_j^m \cap S^1,$$

for all $m \geq n_0$, where n_0 is the constant from (G 6).

Proof. Note first that $[\alpha^n]_{k,w} \subset \bigcup_j G_j^m$ for all $k \geq m$ by construction (Theorem 5.10 and (Lⁿ 6)).

Every boundary point of $\bigcup_j G_j^m \cap S^1$ is a point $\tilde{\alpha}^m \in \mathbf{A}^m$ not equivalent (with respect to \sim^w) to α^n . The statement follows from (G 6). \square

Lemma 8.10. *Let \mathcal{F} be a bounded component of the Fatou set of P_w , $\alpha^n \in \mathbf{A}^n$ such that $\sigma(\alpha^n) \in \partial\mathcal{F}$. The gaps $G_j^m = G_j^m(\alpha^n) \in \mathbf{G}_w^m$ are the ones intersecting $[\alpha^n]_{m,w}$ as before. Then*

$$\sigma^{-1}(\partial\mathcal{F}) = \bigcap_m \bigcup_j G_j^m \cap S^1.$$

Proof. The right hand side of the above expression is compact and contains all points of $[\alpha^n]_{\infty,w}$. Since the set $\sigma([\alpha^n]_{\infty,w})$ is dense in $\partial\mathcal{F}$ (Poi 2), it follows that $\sigma(\bigcap_m \bigcup_j G_j^m \cap S^1) \supset \partial\mathcal{F}$. Note that

$$\bigcap_m \bigcup_j G_j^m = \bigcup [(G^n)],$$

where the union on the right hand side is taken over all sequences of white gaps $G^1 \supset G^2 \supset \dots$ such that $G^n \cap [\alpha^n]_{\infty,w} \neq \emptyset$ for all n . For each such sequence the point $\sigma([(G^n)])$ is an accumulation point of $\sigma([\alpha^n]_{\infty,w})$, thus

$$\sigma\left(\bigcap_m \bigcup_j G_j^m \cap S^1\right) = \partial\mathcal{F}.$$

Consider a gap $\tilde{G}^m \in \mathbf{G}_w^m$ that is distinct from all gaps G_j^m . From Lemma 8.9 it follows that $\tilde{G}^m \cap S^1$ and $G_j^{m+n_0} \cap S^1$ are disjoint. By (Poi 3) it follows that these sets are mapped to disjoint sets by σ , thus $\sigma(\tilde{G}^m \cap S^1) \cap \partial\mathcal{F} = \emptyset$. This proves the claim. \square

The same argument as above applies to Julia-type equivalence classes (see Lemma 8.4 and Lemma 8.6 (1)).

Corollary 8.11. *Let $[\alpha^n]_{n,w}$ ($\alpha^n \in \mathbf{A}^n$) be of Julia-type, $\sigma(\alpha^n) = v$. Then*

$$\sigma^{-1}(v) = \bigcup [(G^m)],$$

where the (finite) union is taken over all sets $[(G^m)]$ such that $G^m \cap [\alpha^n]_{m,w} \neq \emptyset$ for all m .

Proposition 8.8 now follows using (G 6).

8.7. The equivalence relation $\approx^{\mathcal{F},w}$. We consider the equivalence relation $\approx^{\mathcal{F},w}$ obtained from the Carathéodory semi-conjugacy, together with the identification of Fatou components as in (8.1). Our main objective is to show the following.

Proposition 8.12. *We have*

$$\approx^{\mathcal{F},w} = \approx^w.$$

Recall that \approx^w is the equivalence relation from Definition 5.13. To prove this proposition some preparation is needed first. Let us first note the following, which is an immediate consequence of Proposition 8.8.

Lemma 8.13. *The relation $\approx^{\mathcal{F},w}$ is an equivalence relation.*

We write $[s]_{\mathcal{F},w} := \{t \in S^1 \mid s \approx^{\mathcal{F},w} t\}$ for equivalence classes of $\approx^{\mathcal{F},w}$. A description of them follows immediately from Section 8.6.

Lemma 8.14. *Consider $[s]_{\mathcal{F},w}$ ($s \in S^1$). Either*

- $[s]_{\mathcal{F},w} \cap \mathbf{A}^\infty = \emptyset$, then

$$[s]_{\mathcal{F},w} = [(G^m)],$$

for one sequence $G^1 \supset G^2 \supset \dots$ of white gaps. Note that in this case

$$[s]_{\mathcal{F},w} \text{ is compactly contained in } G^n \text{ for all } n.$$

- Or there is $\alpha^n \in [s]_{\mathcal{F},w}$, $\alpha^n \in \mathbf{A}^n$. Then

$$[s]_{\mathcal{F},w} = [\alpha^n]_{\mathcal{F},w} = \bigcup [(G^m)],$$

where the union is taken over all sets $[(G^m)]$ (as in (8.5)), satisfying $G^m \cap [\alpha^n]_{m,w} \neq \emptyset$ for all m . Again

$$[s]_{\mathcal{F},w} \text{ is compactly contained in } \bigcup_j G_j^m,$$

for all m , where the union is taken over all white m -gaps intersecting $[\alpha^n]_{m,w}$.

Lemma 8.15. *The equivalence relation $\approx^{\mathcal{F},w}$ is closed.*

Proof. Consider a convergent sequence $s_n \rightarrow s$ in S^1 . Let $t_n \in [s_n]_{\mathcal{F},w}$ for all n , such that $t_n \rightarrow t$. We want to show that $s \approx^{\mathcal{F},w} t$, i.e., $t \in [s]_{\mathcal{F},w}$. This is clearly the case when $s_n = s$ is constant, since $[s]_{\mathcal{F},w}$ is compact.

Thus we can assume that $s_n \neq s$ for all n . Fix an m . Since \mathbf{A}^m is a finite set it follows that $[s_n]_{\mathcal{F},w} \cap \mathbf{A}^m = \emptyset$ for sufficiently large n . Thus we can assume that $[s_n]_{\mathcal{F},w} \cap \mathbf{A}^n = \emptyset$, for all n . It follows that

$$[s_n]_{\mathcal{F},w} \text{ is contained in a single white } n\text{-gap } G^n.$$

Assume first that $[s]_{\mathcal{F},w}$ contains $\alpha^n \in \mathbf{A}^n$. Let $G_j^m \in \mathbf{G}_w^m$ be the gaps intersecting $[\alpha^n]_{m,w}$ as in Lemma 8.14. Since $[s]_{\mathcal{F},w}$ is compactly contained in $\bigcup_j G_j^m$ it follows that s_n is in the interior of $\bigcup_j G_j^m$ for sufficiently large n . Taking a subsequence if necessary as before, we can assume that

$$s_m \text{ is in the interior of } \bigcup_j G_j^m,$$

for all m . Thus the white m -gap $G^m \supset [s_m]_{\mathcal{F},w}$ equals one of the gaps G_j^m . It follows that $t \in \bigcap_m \bigcup_j G_j^m = [s]_{\mathcal{F},w}$ as desired.

The case when $[s]_{\mathcal{F},w}$ contains no angle in \mathbf{A}^∞ is proved by exactly the same argument. \square

Proof of Proposition 8.12. From the second part in Lemma 8.14 it follows that $[\alpha^n]_{n,w} \subset [\alpha^n]_{\mathcal{F},w}$ for all $\alpha^n \in \mathbf{A}^n$. Thus

$$\sim^{n,w} \leq \overset{\mathcal{F},w}{\approx}$$

for all n .

Claim. $s, t \in [(G^n)]$ implies $s \overset{w}{\sim} t$.

Recall from (G 2) that $G^n \cap S^1$ consists of k n -arcs $[\alpha_0^n, \beta_0^n], \dots, [\alpha_{k-1}^n, \beta_{k-1}^n]$, where $\beta_j^n \overset{n,w}{\sim} \alpha_{j+1}^n$ (lower index taken mod k). Since $\lim_n \alpha_j^n = \lim_n \beta_j^n$ for all j the claim follows.

Thus

$$\overset{\infty,w}{\sim} \leq \overset{\mathcal{F},w}{\approx} \leq \overset{w}{\sim}.$$

Taking the closures (as in Lemma 4.5) yields the result by Lemma 8.15. \square

8.8. The black polynomial P_b . The *black polynomial* P_b is the one obtained from Poirier's Theorem from the black critical portrait, i.e., the sets $[\alpha^1]_{1,b}$ (for all $\alpha^1 \in \mathbf{A}^1$). More precisely P_b is the (unique monic, centered, postcritically finite) polynomial such that the equivalence relation defined by (for all $s, t \in \mathbb{R}/\mathbb{Z}$)

$$s \overset{b}{\approx} t : \Leftrightarrow \sigma_b(-s) = \sigma_b(-t)$$

is equal to the equivalence relation $\overset{b}{\approx}$ defined in terms of the black gaps as in Section 8.3. Here σ_b is a Carathéodory semi-conjugacy of the Julia set of P_b .

The equivalence relation $\overset{\mathcal{F},b}{\approx}$ on S^1 is then defined as in (8.2). As in Proposition 8.12 it follows that

$$\overset{\mathcal{F},b}{\approx} = \overset{b}{\sim},$$

where $\overset{b}{\sim}$ was defined in Definition 5.13.

8.9. Proof of Theorem 1.5. We assume now that F has no periodic critical points. This means that there are no Fatou-type equivalence classes of $\overset{1,w}{\sim}, \overset{1,b}{\sim}$ (Proposition 8.2), hence no Fatou-type classes of $\overset{n,w}{\sim}, \overset{n,b}{\sim}$. The white polynomial P_w is defined as in Section 8.4, the black polynomial P_b as in Section 8.8.

From (Poi 2) it follows that the Fatou sets of P_w, P_b have no bounded components, thus their Julia sets are dendrites. Let $\overset{w}{\approx} = \overset{w}{\sim}, \overset{\mathcal{F},w}{\approx}$ be the equivalence relations from Section 8.3 and Section 8.7. Then $\overset{b}{\approx}, \overset{\mathcal{F},b}{\approx}$ are defined analogously in terms of the black equivalence relations $\overset{n,b}{\sim}$. Since P_w, P_b have no bounded Fatou components it follows from Proposition 8.12 that

$$\overset{w}{\approx} = \overset{\mathcal{F},w}{\approx} = \overset{w}{\sim} \quad \text{and} \quad \overset{b}{\approx} = \overset{\mathcal{F},b}{\approx} = \overset{b}{\sim}.$$

Recall that \sim is the equivalence relation (on S^1) induced by the invariant Peano curve (1.1).

Lemma 8.16. *If F has no critical periodic points, then*

$$\overset{w}{\sim} \vee \overset{b}{\sim} = \sim.$$

Proof. We need to show that $\overset{w}{\sim} \vee \overset{b}{\sim}$ is closed. Let $s \overset{w}{\sim} \vee \overset{b}{\sim} t$. Then

$$s = s_1 \overset{w}{\sim} s_2 \overset{b}{\sim} \dots \overset{w}{\sim} s_{M-1} \overset{b}{\sim} s_M = t,$$

for some points $s_1, \dots, s_M \in S^1$. Since $\overset{w}{\sim} \vee \overset{b}{\sim} \leq \sim$ and the size of each equivalence class with respect to \sim is at most N (Theorem 6.1); it follows that we can choose $M = 2N$. The argument now is identical to the one in the proof of Proposition 8.7. \square

Theorem 1.5 now follows using Theorem 1.3 and Lemma 8.1.

9. PROOF OF THEOREM 1.6

We finish the proof of Theorem 1.6 here. The white/black polynomials P_w, P_b are defined as in Section 8.4 and 8.8.

Recall from (8.1), (8.2) the definition of the associated equivalence relations $\overset{\mathcal{F},w}{\approx}, \overset{\mathcal{F},b}{\approx}$. In Proposition 8.12 it was shown that $\overset{\mathcal{F},w}{\approx} = \overset{w}{\sim}$ as well as $\overset{\mathcal{F},b}{\approx} = \overset{b}{\sim}$. Let \sim be the closure of $\overset{\mathcal{F},w}{\approx} \vee \overset{\mathcal{F},b}{\approx} = \overset{w}{\sim} \vee \overset{b}{\sim}$. Clearly \sim is the equivalence relation induced by the invariant Peano curve γ . Recall the definition of $P_w \hat{\Pi} P_b$ from (1.6). In Section 9.2 we will show the following lemma.

Lemma 9.1. *The map*

$$P_w \hat{\Pi} P_b : \mathcal{K}_w \hat{\Pi} \mathcal{K}_b \rightarrow \mathcal{K}_w \hat{\Pi} \mathcal{K}_b$$

is well defined and topologically conjugate to

$$z^d / \sim : S^1 / \sim \rightarrow S^1 / \sim.$$

Theorem 1.6 follows, using Theorem 1.3.

9.1. Closures. Here we collect some elementary lemmas that will be needed.

Let S, S' be compact metric spaces, $h: S \rightarrow S'$ be a continuous surjection, and \approx be an equivalence relation on S' . The equivalence relation \sim on S defined by

$$s \sim t :\Leftrightarrow h(s) \approx h(t), \text{ for all } s, t \in S,$$

is called the *pullback* of \approx by h .

Lemma 9.2. *In the setting as above, \sim is closed if and only if \approx is closed.*

The proof is straightforward and left as an exercise.

Lemma 9.3. *In the setting as in the last lemma, let $\hat{\sim}$ be the closure of \sim , $\hat{\approx}$ the closure of \approx . Then $\hat{\sim}$ is the pullback of $\hat{\approx}$ by h .*

Proof. Since $\hat{\sim}$ is the *smallest* closed equivalence relation bigger than \sim , it follows from Lemma 9.2 that the pullback of $\hat{\approx}$ by h is bigger than $\hat{\sim}$.

We need to show that $\hat{\sim}$ is a pullback. Define

$$s' \simeq t' :\Leftrightarrow \text{there exists } s, t \in S \text{ such that } s \hat{\sim} t \text{ and } s' = h(s), t' = h(t),$$

for all $s', t' \in S'$. It is straightforward to check that this is an equivalence relation.

Claim. $\hat{\sim}$ is the pullback of \simeq by h .

Let $s \hat{\sim} t$, then by definition $h(s) \simeq h(t)$, thus they are equivalent with respect to the pullback of \simeq by h .

Let s, t be equivalent with respect to the pullback of \simeq by h . Thus $h(s) \simeq h(t)$, meaning there exists $\tilde{s}, \tilde{t} \in S$ such that $h(\tilde{s}) = h(s)$, $h(\tilde{t}) = h(t)$ and $\tilde{s} \hat{\sim} \tilde{t}$. Thus

$$s \sim \tilde{s} \hat{\sim} \tilde{t} \sim t \Rightarrow s \hat{\sim} t,$$

proving the claim.

Let $s', t' \in S'$ such that $s' \approx t'$. Then $s \sim t$ for all $s, t \in S$ satisfying $h(s) = s'$, $h(t) = t'$. Thus $s \hat{\sim} t$ (for all such s, t), which implies $s' \simeq t'$. This means that

$$\simeq \geq \hat{\sim}.$$

It follows that $\simeq \geq \hat{\approx}$, since \simeq is closed by Lemma 9.2. Thus $\hat{\sim}$ is bigger than the pullback of $\hat{\approx}$, finishing the proof. \square

Consider now a continuous surjection $\mu: S \rightarrow S$ on a compact metric space S . An equivalence relation \sim on S is called *invariant* with respect to μ if

$$s \sim t \Rightarrow \mu(s) \sim \mu(t) \text{ for all } s, t \in S.$$

Lemma 9.4. *Let $\mu: S \rightarrow S$ be continuous, surjective; \sim an equivalence relation on S invariant with respect to μ . Then the closure $\hat{\sim}$ of \sim is invariant with respect to μ .*

Proof. Consider the equivalence relation \approx given by

$$s \approx t :\Leftrightarrow \mu(s) \hat{\sim} \mu(t),$$

for all $s, t \in S$. From Lemma 9.2 it follows that \approx is closed. Note that

$$s \sim t \Rightarrow \mu(s) \sim \mu(t) \Rightarrow \mu(s) \hat{\sim} \mu(t) \Rightarrow s \approx t,$$

meaning that $\approx \geq \sim$. Note that the meet of any two closed equivalence relations is closed (see (4.2)), thus the meet

$$\begin{aligned} \approx \wedge \hat{\sim} & \text{ is closed and bigger than } \sim; \text{ which implies} \\ \hat{\sim} & = \approx \wedge \hat{\sim}. \end{aligned}$$

Thus for all $s, t \in S$

$$s \hat{\sim} t \Rightarrow s \approx t \Rightarrow \mu(s) \hat{\sim} \mu(t),$$

finishing the proof. \square

In the next lemma we “take the closure of a commutative diagram and show that everything goes well”. Let \sim, \approx be equivalence relations on compact metric spaces S, S' . The maps $\mu: S \rightarrow S, \varphi: S' \rightarrow S'$ as well as $h: S \rightarrow S'$ are continuous surjections such that the following diagram commutes

$$\begin{array}{ccc} (S, \sim) & \xrightarrow{\mu} & (S, \sim) \\ \downarrow h & & \downarrow h \\ (S', \approx) & \xrightarrow{\varphi} & (S', \approx). \end{array}$$

By this is meant that $\varphi \circ h = h \circ \mu$. The equivalence relation \sim is invariant with respect to μ , and \approx is invariant with respect to φ . Furthermore \sim is the pullback of \approx by h .

Lemma 9.5. *In the setting as above, let $\hat{\sim}$ be the closure of \sim ; $\hat{\approx}$ be the closure of \approx . Then*

$$\mu/\hat{\sim}: S/\hat{\sim} \rightarrow S/\hat{\sim}$$

is topologically conjugate to

$$\varphi/\hat{\approx}: S'/\hat{\approx} \rightarrow S'/\hat{\approx}.$$

Proof. We note first that the maps $\mu/\hat{\sim}, \varphi/\hat{\approx}$ are well defined by Lemma 9.4. From Lemma 9.3 it follows that $\hat{\sim}$ is the pullback of $\hat{\approx}$ by h . From (CE 4) it follows that $S'/\hat{\approx}$ is a compact Hausdorff space. Applying Lemma 4.6 to the map $S \xrightarrow{h} S' \rightarrow S'/\hat{\approx}$ yields that $S/\hat{\sim}$ is homeomorphic to $S'/\hat{\approx}$, where the homeomorphism is given by $\tilde{h}([s]_{\hat{\sim}}) = [h(s)]_{\hat{\approx}}$. Write $\tilde{\varphi} = \varphi/\hat{\approx}, \tilde{\mu} = \mu/\hat{\sim}$, then

$$\tilde{\varphi} \circ \tilde{h}([s]_{\hat{\sim}}) = \tilde{\varphi}([h(s)]_{\hat{\approx}}) = [\varphi \circ h(s)]_{\hat{\approx}} = [h \circ \mu(s)]_{\hat{\approx}} = \tilde{h}([\mu(s)]_{\hat{\sim}}) = \tilde{h} \circ \tilde{\mu}([s]_{\hat{\sim}}).$$

This finishes the proof. \square

9.2. Proof of Lemma 9.1. We first show that $P_w \hat{\Pi} P_b$ is well defined.

Consider $\mathcal{K}_w \sqcup \mathcal{K}_b$, the disjoint union of $\mathcal{K}_w, \mathcal{K}_b$. The equivalence relation \sim on $\mathcal{K}_w \sqcup \mathcal{K}_b$ is the one generated by

$$(9.1) \quad \begin{aligned} \sigma_w(t) & \sim \sigma_b(-t), \quad \text{for all } t \in \mathbb{R}/\mathbb{Z} \text{ and} \\ x & \sim y, \quad \text{if } x, y \in \text{clos } \mathcal{F}_w \text{ or } x, y \in \text{clos } \mathcal{F}_b, \end{aligned}$$

for all $x, y \in \mathcal{K}_w$, or $x, y \in \mathcal{K}_b$. Here $\mathcal{F}_w, \mathcal{F}_b$ are bounded components of the Fatou sets of P_w, P_b and $\sigma_w(t) \in \mathcal{K}_w \subset \mathcal{K}_w \sqcup \mathcal{K}_b, \sigma_b(-t) \in \mathcal{K}_b \subset \mathcal{K}_w \sqcup \mathcal{K}_b$. The map

$$P_w \sqcup P_b: \mathcal{K}_w \sqcup \mathcal{K}_b \rightarrow \mathcal{K}_w \sqcup \mathcal{K}_b,$$

is well defined. Furthermore the equivalence relation defined in (9.1) is invariant with respect to $P_w \sqcup P_b$. Let $\hat{\sim}$ be the closure of \sim , and $\mathcal{K}_w \hat{\Pi} \mathcal{K}_b = \mathcal{K}_w \sqcup \mathcal{K}_b / \hat{\sim}$. From Lemma 9.4 it follows that $P_w \sqcup P_b$ descends to this quotient, meaning that

$$P_w \hat{\Pi} P_b : \mathcal{K}_w \hat{\Pi} \mathcal{K}_b \rightarrow \mathcal{K}_w \hat{\Pi} \mathcal{K}_b,$$

is well defined.

We first show a one-sided version of Lemma 9.1. Identify the closure of each bounded Fatou component in \mathcal{K}_w to form the quotient $\mathcal{K}_w / \mathcal{F}$. Recall that the Fatou set of P_w is separated. Since P_w maps each bounded Fatou component to a bounded Fatou component the quotient map

$$P_w / \mathcal{F} : \mathcal{K}_w / \mathcal{F} \rightarrow \mathcal{K}_w / \mathcal{F}$$

is well defined.

Lemma 9.6. *The map P_w / \mathcal{F} as above is topologically conjugate to*

$$z^d : S^1 / \overset{\mathcal{F}, w}{\approx} \rightarrow S^1 / \overset{\mathcal{F}, w}{\approx}.$$

Proof. Consider the equivalence relation on \mathcal{K}_w , defined by $(x, y \in \mathcal{K}_w)$

$$x \overset{\mathcal{F}}{\approx} y :\Leftrightarrow x, y \in \text{clos } \mathcal{F},$$

where \mathcal{F} is a bounded Fatou component of P_w .

Claim. $\overset{\mathcal{F}}{\approx}$ is closed.

Consider two convergent sequences $x_n \rightarrow x_0, y_n \rightarrow y_0$ in \mathcal{K}_w , satisfying $x_n \overset{\mathcal{F}}{\approx} y_n$ (for all $n \geq 1$). We need to show that $x_0 \overset{\mathcal{F}}{\approx} y_0$. This is clear when the sequence (x_n) is contained in a single equivalence class of $\overset{\mathcal{F}}{\approx}$.

Assume now that each x_n is contained in a distinct equivalence class, which we can assume to be non-trivial. This means that x_n, y_n are contained in the closure of the same bounded Fatou component \mathcal{F}_n . From the subhyperbolicity of P_w it follows that $\text{diam } \mathcal{F}_n \rightarrow 0$, thus $x_0 = \lim x_n = \lim y_n = y_0$ proving the claim.

From the claim it follows, that $\mathcal{K}_w / \mathcal{F}$ is a compact Hausdorff space (see (CE 4)). Clearly $\overset{\mathcal{F}, w}{\approx}$ is the equivalence relation (on S^1) induced by the map $S^1 \xrightarrow{\sigma_w} \mathcal{K}_w \rightarrow \mathcal{K}_w / \mathcal{F}$. Thus we obtain from Lemma 4.6 that $S^1 / \overset{\mathcal{F}, w}{\approx}$ is homeomorphic to $\mathcal{K}_w / \mathcal{F}$. The topological conjugacy is clear, since P_w maps each point $\sigma_w(t) \in \mathcal{K}_w$ to $\sigma_w(dt) \in \mathcal{K}_w$ (for all $t \in \mathbb{R}/\mathbb{Z}$). \square

Lemma 9.7. *The equivalence relation $\overset{w}{\sim} \vee \overset{b}{\sim}$ is invariant with respect to μ .*

Proof. From $(\mathcal{L}^n 4)$ it follows that $\overset{\infty, w}{\sim} = \bigvee \overset{n, w}{\sim}$ is invariant with respect to μ . Thus $\overset{w}{\sim}$ (being the closure of $\overset{\infty, w}{\sim}$) is invariant with respect to μ by Lemma 9.4. Similarly $\overset{b}{\sim}$ is μ -invariant. It is immediate that the join of two invariant equivalence relations is invariant. \square

Proof of Lemma 9.1. Let $\mathcal{K}_w / \mathcal{F}$ be as in the last lemma, $\mathcal{K}_b / \mathcal{F}$ the quotient obtained by identifying bounded Fatou component of \mathcal{K}_b . Consider the equivalence

relation \simeq on $\mathcal{K}_w/\mathcal{F} \sqcup \mathcal{K}_b/\mathcal{F}$ generated by

$$\begin{aligned} [\sigma_w(t)] &\simeq [\sigma_b(-t)], \text{ where} \\ [\sigma_w(t)] &\in \mathcal{K}_w/\mathcal{F} \subset \mathcal{K}_w/\mathcal{F} \sqcup \mathcal{K}_b/\mathcal{F} \\ [\sigma_b(-t)] &\in \mathcal{K}_b/\mathcal{F} \subset \mathcal{K}_w/\mathcal{F} \sqcup \mathcal{K}_b/\mathcal{F}, \end{aligned}$$

for all $t \in \mathbb{R}/\mathbb{Z}$. Clearly \simeq is invariant with respect to $P_w/\mathcal{F} \sqcup P_b/\mathcal{F}$. Consider the map

$$S^1 \sqcup S^1 \xrightarrow{[\sigma_w(s)], [\sigma_b(-t)]} \mathcal{K}_w/\mathcal{F} \sqcup \mathcal{K}_b/\mathcal{F}.$$

The pullback of \simeq is $\overset{w}{\sim} \vee \overset{b}{\sim}$ (on each S^1), see Proposition 8.12; it is invariant with respect to μ (Lemma 9.7). Thus we have the following commutative diagram

$$\begin{array}{ccc} (S^1 \sqcup S^1, \overset{w}{\sim} \vee \overset{b}{\sim}) & \xrightarrow{\mu} & (S^1 \sqcup S^1, \overset{w}{\sim} \vee \overset{b}{\sim}) \\ \downarrow & & \downarrow \\ (\mathcal{K}_w/\mathcal{F} \sqcup \mathcal{K}_b/\mathcal{F}, \simeq) & \longrightarrow & (\mathcal{K}_w/\mathcal{F} \sqcup \mathcal{K}_b/\mathcal{F}, \simeq). \end{array}$$

The quotient of $\mathcal{K}_w/\mathcal{F} \sqcup \mathcal{K}_b/\mathcal{F}$ with respect to the closure $\hat{\simeq}$ is $\mathcal{K}_w \hat{\Pi} \mathcal{K}_b$; the quotient of the map $P_w/\mathcal{F} \sqcup P_b/\mathcal{F}$ is $P_w \hat{\Pi} P_b$ (see (1.6)).

The closure of $\overset{w}{\sim} \vee \overset{b}{\sim}$ is \sim (the equivalence relation induced by γ). Clearly $S^1 \sqcup S^1 / \sim = S^1 / \sim$. Note that $\mathcal{K}_w/\mathcal{F}, \mathcal{K}_b/\mathcal{F}$ are metrizable ([Dav86, Proposition 2.2]). The claim follows from Lemma 9.5. \square

The author believes that in general $\overset{w}{\sim} \vee \overset{b}{\sim} \neq \sim$, in particular $\overset{w}{\sim} \vee \overset{b}{\sim}$ will not be closed in general. We do not present the examples that seem to indicate this here.

10. FRACTAL TILINGS

From the invariant Peano curve $\gamma: S^1 \rightarrow S^2$ one obtains *fractal tilings*. Indeed divide the circle \mathbb{R}/\mathbb{Z} in d intervals $[j/d, (j+1)/d]$ ($j = 0, \dots, d-1$). Since μ maps each such interval onto \mathbb{R}/\mathbb{Z} it follows from Theorem 1.1 that F maps each set $\gamma([j/d, (j+1)/d])$ to the whole sphere. The tiling lifts to the *orbifold covering* (which is either the Euclidean or the hyperbolic plane). The thus obtained tiles are illustrated for the example from Section 5.1 in Figure 3.

This example is atypical however, since usually the tiles are very fractal. We show the fractal tilings obtained from the Peano curve γ for two more examples.

The first is a Lattès map whose orbifold has signature $(2, 3, 6)$. It is the map $R_4 = 1 - (3z+1)^3/(9z-1)^2$ from [Mey02, Section 6.1]. The first approximation γ^1 of the Peano curve is illustrated (in the orbifold covering) in Figure 4. Tiles given by the resulting Peano curve are illustrated in Figure 5. The two critical portraits (i.e., the equivalence classes of $\overset{1,w}{\sim}, \overset{1,b}{\sim}$) that describe R_4 according to Theorem 1.4 are:

$$\begin{aligned} \text{white portrait: } & \left\{ \frac{1}{9}, \frac{4}{9}, \frac{7}{9} \right\}, \\ \text{black portrait: } & \left\{ \frac{1}{3}, \frac{2}{3} \right\}, \left\{ \frac{1}{6}, \frac{5}{6} \right\}. \end{aligned}$$

The second example is the map $R_2 = 1 - 2(z-1)(z+3)^3/((z+1)(z-3)^3)$ (see [Mey02, Section 6.1]). It is a Lattès map whose orbifold has signature $(3, 3, 3)$. The

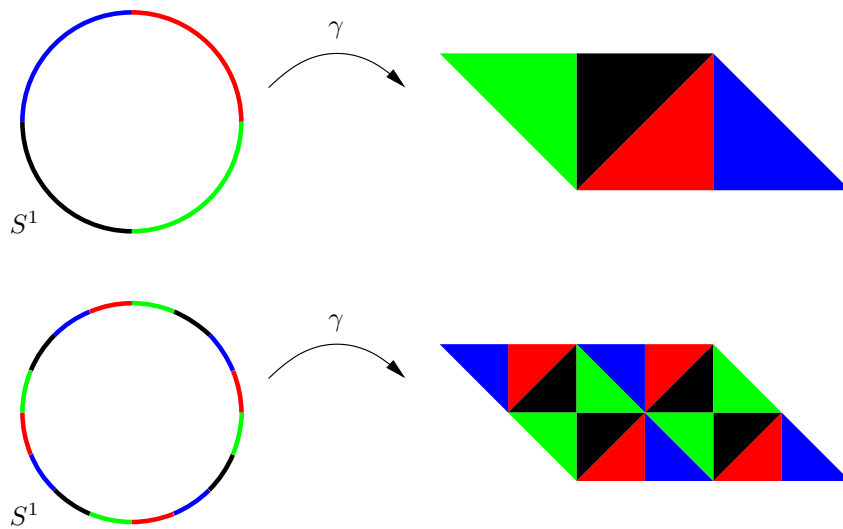
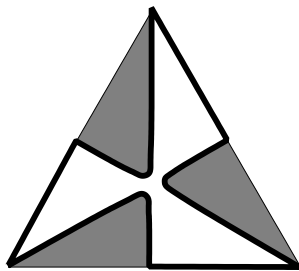
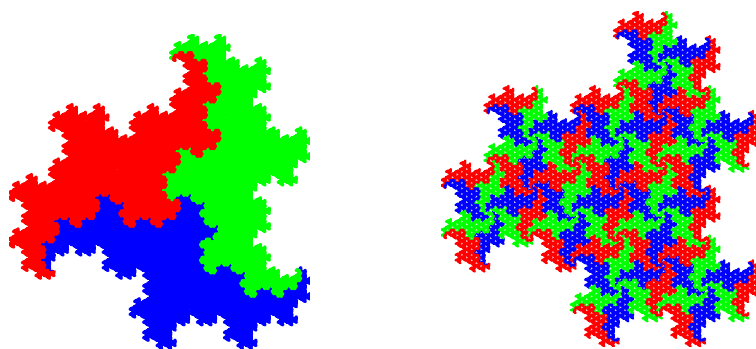


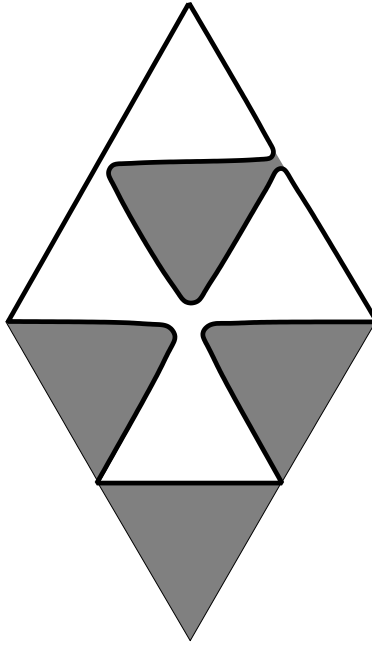
FIGURE 3. Tiling given by example from Section 5.1

FIGURE 4. First approximation γ^1 for R_4 .

(a) First order Tiles.

(b) Forth order tiles.

FIGURE 5. Tilings induced by the Peano curve of R_4 .

FIGURE 6. First approximation γ^1 for R_2 .

first approximation γ^1 is shown (in the orbifold covering) in Figure 6. The tiles that are obtained from the resulting invariant Peano curve are shown in Figure 7. The two critical portraits describing R_2 are:

$$\begin{aligned} \text{white portrait: } & \left\{ \frac{7}{60}, \frac{22}{60}, \frac{37}{60} \right\}, \left\{ \frac{43}{60}, \frac{58}{60} \right\}, \\ \text{black portrait: } & \left\{ \frac{15}{60}, \frac{30}{60}, \frac{45}{60} \right\}, \left\{ \frac{13}{60}, \frac{58}{60} \right\}. \end{aligned}$$

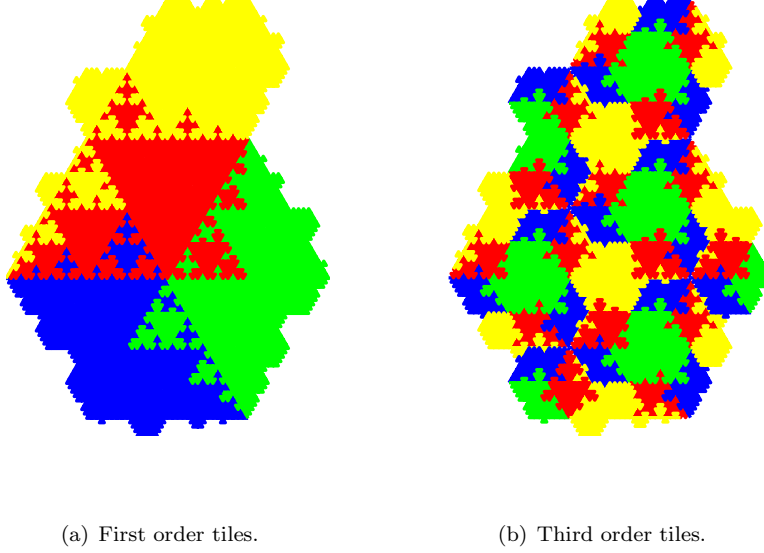
All examples considered above had *parabolic orbifold*. Consider a *rational expanding Thurston map* (meaning it has no *Thurston obstruction*) with *hyperbolic orbifold*. The tiling obtained from the invariant Peano curve lifts to the *orbifold cover*, i.e., the hyperbolic plane. Thus one obtains fractal tilings of the hyperbolic plane with interesting self-similar properties.

There are other ways to obtain fractal tilings from the invariant Peano curve γ . Instead of dividing the circle into d^n intervals of the same length, we can take the images of the n -arcs by γ . Thus we get tilings of the hyperbolic/Euclidean plane with $k(= \# \text{ post})$ different tiles. Each tile divides into tiles of the $(n+1)$ -th order.

There is yet another way to obtain tilings from the invariant Peano curve γ in a natural way. Namely define tiles as the images of (either white or black) n -gaps by γ .

11. OPEN QUESTIONS

The first question to ask is whether the construction of the mating presented here is the general case; at least when there are no periodic critical points.

FIGURE 7. Tilings induced by the Peano curve of R_2 .

Open Problem 1. Let F be an expanding Thurston map that does not have periodic critical points. Assume further that F is obtained as a mating of two polynomials. Is the *mating always obtained as constructed here* (as well as in [Mey])? In particular this would mean that there is a pseudo-isotopy H^0 as in [Mey, Section 3.1].

In [Mey, Section 8] it was shown that a certain Lattès map h does not admit a pseudo-isotopy H^0 . Thus it cannot be shown from the results in the paper present that h is obtained as a mating (an iterate however is). This means a natural candidate for an expanding Thurston map that does not arise as a mating is h .

Several people have asked whether there is a bound on the number of points that are identified in a mating. If a matings (of strictly preperiodic polynomials) is obtained as constructed here this is answered by Theorem 6.1.

Open Problem 2. Is it possible to decide whether an expanding Thurston map F is equivalent to a rational map from the critical portraits (see Section 5.5)? By Thurston’s topological characterization [DH93] this amounts to the question whether it is possible to read off *Thurston obstructions* from the *critical portraits*.

In principle this is possible. Recall that each 1-tile/1-edge has a natural corresponding 1-gap/1-arc. Thus every multicurve in $S^2 \setminus \text{post}$ can be naturally represented in the “critical portrait sphere” \tilde{S}^2 (i.e., the sphere whose two hemispheres are S_w^2 and S_b^2 as in Section 5). A multicurve Γ in this picture is just a multicurve in $\tilde{S}^2 \setminus \mathbf{A}^0$. Since each 1-gap is mapped by μ to S_w^2 or S_b^2 , it is possible to take the preimage of the multicurve. It is invariant if each component of the preimage is isotopic rel. \mathbf{A}^0 to one component of Γ . The Thurston matrix is then taken as usual.

However it is not clear whether the description above offers any advantage in finding Thurston obstructions.

Open Problem 3. Consider a postcritically finite rational map \tilde{f} whose *Julia set* is a *Sierpiński carpet*. Identifying the closure of each Fatou component yields an expanding Thurston map f (see Section 2.1). Assume f has an invariant Peano curve γ (meaning we do not have to take an iterate in Theorem 1.1). Is it possible to construct from γ a semiconjugacy $\tilde{\gamma}: S^1 \rightarrow \tilde{\mathcal{J}}$ ($\tilde{\mathcal{J}}$ is the Julia set of \tilde{f}) such that $\tilde{f}(\tilde{\gamma}(z)) = \tilde{\gamma}(z^d)$ for all $z \in S^1$ (where $d = \deg \tilde{f}$)? This is false in general (see [Kam03, Section 4]), but possibly true under some additional assumptions.

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